MA 16020 Lesson 34: Determinants

The **determinant** of a square matrix A is a certain number assigned to the matrix, $\det(A)$.

Important properties:

1) (multiplicativity) $\det(A \cdot B) = \det(A) \cdot \det(B)$

2) (det of unit matrix) $\det(I_n) = 1$

3) (det and invertibility) A is invertible iff and only if $\det(A) \neq 0$

(if $\det(A) = 0$, we call A singular)

Determinant of a 1×1 **matrix** is "itself": $\det[a] = a$

Determinant of a 2×2 **matrix** is computed as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} =: \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot d - b \cdot c$$

Examples:

$$\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} = 3 \cdot 2 - 1 \cdot (-5) = 6 + 5 = 11$$

$$\begin{vmatrix} -3 & -4 \\ 2 & 3 \end{vmatrix} = (-3) \cdot 3 - (-4) \cdot 2 = -9 + 8 = -1$$

$$\begin{vmatrix} 1 & 4 \\ -2 & -8 \end{vmatrix} = 1 \cdot (-8) - 4 \cdot (-2) = -8 - (-8) = 0$$

singular / non-invertible

Determinants of 3×3 matrices:
(A) **directly:**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot e \cdot i + d \cdot h \cdot c + g \cdot b f - c \cdot e \cdot g - f \cdot h \cdot a - i b d$$

$$\begin{array}{ccc} a & b & c \\ d & e & f \end{array}$$

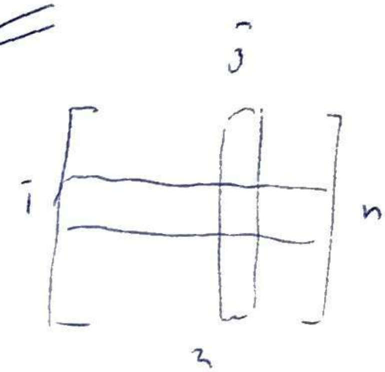
Example:

$$\begin{vmatrix} 1 & 3 & 2 \\ 4 & -2 & -1 \\ 6 & 3 & 2 \end{vmatrix} = 1 \cdot (-2) \cdot 2 + 4 \cdot 3 \cdot 2 + 6 \cdot 3 \cdot (-1) - 6 \cdot (-2) \cdot 2 - 1 \cdot 3 \cdot (-1) - 4 \cdot 3 \cdot 2$$

$$= -4 + 24 - 18 + 24 + 3 - 24 = 5$$

$$\begin{array}{ccc} 1 & 3 & 2 \\ 4 & -2 & -1 \end{array}$$

(B) **using cofactors** (also works for $n \times n$ matrix, for any n):



For a given square matrix A,

its **minor** M_{ij} is: determinant of the matrix obtained from A by deleting its i-th row and j-th column

and its **cofactor** C_{ij} is: $(-1)^{i+j} \cdot M_{ij}$

Example: For the matrix $\begin{bmatrix} 3 & 1 & 0 \\ -2 & 1 & 5 \\ 4 & 7 & -3 \end{bmatrix}$, one has:

the minor $M_{12} = \det \begin{bmatrix} -2 & 5 \\ 4 & -3 \end{bmatrix} = (-2)(-3) - 5 \cdot 4 = 6 - 20 = -14$

the cofactor $C_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \cdot M_{12} = (-1) \cdot M_{12} = (-1) \cdot (-14) = 14$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Using cofactors, the determinant of A is given as:

$$\det(A) = a_{i1} \cdot C_{i1} + a_{i2} \cdot C_{i2} + \dots a_{in} \cdot C_{in} \quad \text{for any choice of } i,$$

$$\det(A) = a_{1j} \cdot C_{1j} + a_{2j} \cdot C_{2j} + \dots + a_{nj} \cdot C_{nj}, \quad \text{for any choice of } j$$

Example: Compute the determinant of the matrix $\begin{bmatrix} 3 & 1 & 0 \\ -2 & 1 & 5 \\ 4 & 7 & -3 \end{bmatrix}$ by cofactor expansion with respect to the first row.

$$\det(A) = \underbrace{3}_{a_{11}} \cdot \underbrace{(-1)^{1+1} \cdot \begin{vmatrix} 1 & 5 \\ 7 & -3 \end{vmatrix}}_{C_{11}} + \underbrace{1}_{a_{12}} \cdot \underbrace{(-1)^{1+2} \begin{vmatrix} -2 & 5 \\ 4 & -3 \end{vmatrix}}_{C_{12}} + \underbrace{0}_{a_{13}} \cdot \underbrace{(-1)^{1+3} \begin{vmatrix} -2 & 1 \\ 4 & 7 \end{vmatrix}}_{C_{13}}$$

$$= 3\big((-3) - 35\big) + (-1)(6 - 20) + 0 =$$

$$= 3 \cdot (-38) - (-14) = -3 \cdot 38 + 14 = -100$$

Inverse matrix via cofactors. The **cofactor matrix** $C(A)$ is a matrix that has on position (i, j) the cofactor C_{ij}.

The **adjugate matrix** $\mathrm{Adj}(A)$ is the "cofactor matrix transposed": on position (i, j), it has the cofactor C_{ji}.

Ex:
$$C(A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \leadsto \mathrm{Adj}(A) = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$$

If the matrix A is invertible, its inverse can be then computed by the formula:

$$\boxed{A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{Adj}(A)}$$

A particular case: 2×2 matrices. Given an invertible matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse matrix can be found as follows:

$$C(A) = \begin{bmatrix} (-1)^{1+1} d & (-1)^{1+2} c \\ (-1)^{2+1} b & (-1)^{2+2} a \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$Adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{\det A} \cdot Adj(A) = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Example: If it exists, find the inverse of the matrix $\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$.

$$A^{-1} = \frac{1}{4 \cdot 2 - (-3) \cdot 1} \cdot \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} =$$

$$= \frac{1}{11} \cdot \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2/11 & 3/11 \\ -\frac{1}{11} & \frac{4}{11} \end{bmatrix}$$