MA 16020 Lesson 33: Inverse matrices

Recall (Unit matrix): The matrix I (or, more precisely, I_n) denotes the $n \times n$ **identity matrix**

$$I_n = \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{n} \Big\} n$$

It has the properties that 1) for every $n \times k$ matrix B, $I_n \cdot B = B$

2) for every $k \times n$ matrix C, $C \cdot I_n = C$

Inverse matrices. Given a square ($n \times n$) matrix A, its *inverse matrix*, is an ($n \times n$) matrix A^{-1} such that:

$$A^{-1} \cdot A = A \cdot A^{-1} = I_n$$

Example: $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$, and vice versa

Matrix multiplication and systems of equations. Using matrix multiplication, a system of linear equations can be expressed as a single matrix equation.

Example:

The system
$$\begin{aligned} 2x + y + z &= 1 \\ x + 2z &= -2 \\ y - z &= 4 \end{aligned}$$
corresponds to
$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}}_{\text{"coefficient matrix"}} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

The inverse matrix in this case is:

$$\begin{bmatrix} 1 & -1 & -1 \\ -1/2 & 1 & 3/2 \\ -1/2 & 1 & 1/2 \end{bmatrix}$$

We can use the inverse matrix to solve the matrix equation as follows:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \quad / \cdot \begin{bmatrix} 1 & -1 & -1 \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 7/2 \\ -\frac{1}{2} \end{bmatrix} \rightsquigarrow \underline{\underline{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 7/2 \\ -1/2 \end{bmatrix}}}$$

How to compute the inverse matrix. Matrices for which the inverse exist are called **invertible**. If a marix is not inveritible, then it is called **singular**.

Example: If it exists, find the inverse of the matrix $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{(1)}, \quad \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{(2)}$$

We can solve (1) and (2) simultaneously, putting both the RHS (1), (2) to one matrix:

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{cc|cc} 0 & -1 & 3 & -2 \\ 1 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 0 & +1 & -3 & +2 \\ 1 & 0 & 2 & -1 \end{array}\right] \rightsquigarrow \underline{\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right]}$$

$$\rightsquigarrow \text{get } \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \text{ so } \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}}}$$

Finding the inverse of a matrix – summary.

1. For the matrix A to be inverted, write down the augmented matrix:

   $$[A \mid I_n]$$

2. Perform Gauss–Jordan elimination. $[A \mid I_n] \rightsquigarrow [I_n \mid B]$

3. The matrix A^{-1} can be read off from the result as: $A^{-1} = B$

Exercise 1. If it exists, find the inverse of the matrix:

$$A = \begin{bmatrix} -3 & 0 & -9 \\ 2 & 1 & 6 \\ 1 & -2 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -3 & 0 & -9 & 1 & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 \\ 1 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{3}R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -\frac{1}{3} & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 \\ 1 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_3 \to R_3}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -\frac{1}{3} & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 \\ 0 & -2 & 1 & \frac{1}{3} & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 2/3 & 1 & 0 \\ 0 & -2 & 1 & 1/3 & 0 & 1 \end{array}\right] \xrightarrow{2R_2+R_3 \to R_3}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 2/3 & 1 & 0 \\ 0 & 0 & 1 & 5/3 & 2 & 1 \end{array}\right] \xrightarrow{-3R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16/3 & -6 & -3 \\ 0 & 1 & 0 & 2/3 & 1 & 0 \\ 0 & 0 & 1 & 5/3 & 2 & 1 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -16/3 & -6 & -3 \\ 2/3 & 1 & 0 \\ 5/3 & 2 & 1 \end{bmatrix}$$

Exercise 2. Using the inverse of the coefficient matrix, solve the following system of equations:

$$\begin{aligned} 3x - 6y + z &= 2 \\ 2y + 6z &= 3 \\ 2x - 4y + z &= 5 \end{aligned}$$

coefficient matrix $= \begin{bmatrix} 3 & -6 & 1 \\ 0 & 2 & 6 \\ 2 & -4 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 3 & -6 & 1 & 1 & 0 & 0 \\ 0 & 2 & 6 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 2 & 6 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_3 \to R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 2 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 3 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 3 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 0 & 3 \end{array}\right] \xrightarrow{-3R_3+R_2 \to R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 6 & \frac{1}{2} & -9 \\ 0 & 0 & 1 & -2 & 0 & 3 \end{array}\right] \xrightarrow{2R_2+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13 & 1 & -19 \\ 0 & 1 & 0 & 6 & \frac{1}{2} & -9 \\ 0 & 0 & 1 & -2 & 0 & 3 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 & 1 & -19 \\ 6 & \frac{1}{2} & -9 \\ -2 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -66 \\ -\frac{63}{2} \\ 11 \end{bmatrix}$$

Exercise 3. If it exists, find the inverse of the matrix:

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 5 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1 + R_2 \to R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -3 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{array}\right] \xrightarrow{-R_2 + R_3 \to R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -3 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

$$\underbrace{}_{(*)}$$

(*) There is no way to make (*) into an identity matrix → inverse matrix for A does not exist