MA 16020 Lesson 32: Matrix operations

Matrices. An $m \times n$ matrix is: an array (table) of numbers with m rows and n columns

Examples:

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

2×2 matrix, 2×1 matrix, 2×3 matrix, ...

Just as with numbers, there are certain operations for matrices:

Matrix addition (and subtraction). We can add and subtract matrices as long as the dimensions of the matrices agree.
The addition/subtraction is done "component-wise":

Examples:

$$\begin{bmatrix} 6 & -3 & 2 \\ -2 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 4 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 6+2 & -3+1 & 2+0 \\ -2+4 & 4+(-5) & 1+7 \end{bmatrix} = \begin{bmatrix} 8 & -2 & 2 \\ 2 & -1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 8 \\ -3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 1-2 & 8-1 \\ -3-4 & 4-(-5) \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ -7 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -10 \\ -5 & 7 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 1 & 0 \\ 3 & 2 \end{bmatrix} = \text{not defined!}$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 7 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 7 & -3 & 0 \end{bmatrix}$$

Properties of matrix addition. If A, B, C are three $m \times n$ matrices and 0 denotes the $m \times n$ matrix consisting of all zeroes, then:

1. (associativity) $A + (B + C) = (A + B) + C$
2. (commutativity) $A + B = B + A$
3. (neutral element) $0 + A = A = A + 0$
4. ("opposite element") $A + (-A) = (-A) + A = 0$

Scalar multiplication of matrices. We can multiply a matrix of arbitrary dimensions by a number ("scalar"). This is done "component-wise":

Examples:

$$3\begin{bmatrix} 5 & -3 & 1 \\ 2 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 3\cdot 5 & 3\cdot(-3) & 3\cdot 1 \\ 3\cdot 2 & 3\cdot 4 & 3\cdot 9 \end{bmatrix} = \begin{bmatrix} 15 & -9 & 3 \\ 6 & 12 & 27 \end{bmatrix}$$

$$(-1)\begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 3 & 2 \end{bmatrix}$$

$$1\begin{bmatrix} 0 & 5 \\ -2 & -3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -2 & -3 \\ 0 & 4 \end{bmatrix}$$

Properties of scalar multiplication. If A, B are two $m \times n$ matrices and c, d two numbers, then:

1. (associativity) $c\cdot(d\cdot A) = (c\cdot d)\cdot A$
2. (distributivity) $c\cdot(A+B) = c\cdot A + c\cdot B$, $(c+d)\cdot A = c\cdot A + d\cdot A$
3. (unit element) $1\cdot A = A$

Examples:

$$2\begin{bmatrix} 1 & -5 & 3 \\ 7 & 1 & 4 \end{bmatrix} - 3\begin{bmatrix} 2 & 0 & 3 \\ 1 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 2 & -10 & 6 \\ 14 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 9 \\ 3 & 3 & -12 \end{bmatrix} = \begin{bmatrix} -4 & -10 & -3 \\ 11 & -1 & 20 \end{bmatrix}$$

$$2\begin{bmatrix} 1 & 0 & 2 \\ 4 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 & 1 \\ 5 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 8 & 2 & 0 \\ 6 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 15 & 3 & -6 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 7 \\ 23 & 5 & -6 \\ 9 & 7 & 3 \end{bmatrix}$$

$$5\begin{bmatrix} 2 & 0 & 7 \\ -4 & 3 & 2 \\ 1 & 5 & -2 \end{bmatrix} + 4\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} = \text{not defined!}$$

Matrix multiplication. If A is an $m \times n$ matrix and B is an $n \times k$ matrix (so **number of columns of** A**=number of rows of** B), we define the matrix product AB. It is an $m \times k$ matrix, whose

entry on position $(i, j) =$ sum of component-wise product of i-th row of A and j-th column of B

Examples:

result: 1×1 matrix

$$\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + (-1) \cdot 5 + 4 \cdot (-2) \end{bmatrix} = \begin{bmatrix} 2 - 5 - 8 \end{bmatrix} = \begin{bmatrix} -11 \end{bmatrix}$$

result: 2×1 matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 2 \\ 4 \cdot 1 + 5 \cdot 0 + 6 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 + 6 \\ 4 + 12 \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}$$

2×2 matrix

$$\begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + (-2) \cdot 3 & 1 \cdot (-1) + (-2) \cdot 5 \\ 0 \cdot 2 + (-3) \cdot 3 & 0 \cdot (-1) + (-3) \cdot 5 \end{bmatrix} = \begin{bmatrix} -4 & -11 \\ -9 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + (-1) \cdot 0 & 2 \cdot (-2) + (-1) \cdot (-3) \\ 3 \cdot 1 + 5 \cdot 0 & 3 \cdot (-2) + 5 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -21 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot (-2) + 0 \cdot 3 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot (-2) + 1 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix}$$

Properties of matrix multiplication. If A, B, C are matrices of suitable dimensions (i.e. so that the discussed operations are defined), and d a real number, then:

1. (associativity) $A\cdot(B\cdot C) = (A\cdot B)\cdot C$
2. (distributivity) $A\cdot(B+C) = AB + AC$, $(A+B)\cdot C = AC+BC$
3. (unit element) $I\cdot A = A$, $A\cdot I = A$, where $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$
4. (mult. by scalars) $d\cdot(A\cdot B) = (dA)\cdot B = A\cdot(d\cdot B)$

!! We do not have commutativity: $AB \neq BA$ in general!

Exercise. The number of grams of protein and carbohydrates per can of pet food is given by the following table:

	Protein	Carb.
Brand A	15	150
Brand B	13	140
Brand C	14	180

If we mix a meal using one can of brand A, two cans of brand B and a half can of brand C, what will be the overall amount of protein and carohydrates, respectively?

$$\begin{bmatrix} 1 & 2 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 15 & 150 \\ 13 & 140 \\ 14 & 180 \end{bmatrix} = \begin{bmatrix} 15+2\cdot 13+\frac{1}{2}\cdot 14 & 150+2\cdot 140+\frac{1}{2}\cdot 180 \end{bmatrix}$$

$$= \begin{bmatrix} 48 & 520 \end{bmatrix}$$

protein

carbs in the mix