MA 16020 Lesson 31: Gauss–Jordan elimination

Recall (Gaussian elimination): Given a system of linear equations, such as

$$\begin{aligned} 3x + 2y &= 1, \\ x + y &= 1, \end{aligned}$$

we write down its associated *augmented matrix*:

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 1 & 1 \end{array}\right]$$

Then we use the following types of row operations to obtain a matrix in the *row echelon form*:

1. switch rows "$R_i \leftrightarrow R_j$"
2. multiply row by a nonzero constant "$aR_i \rightarrow R_i$"
3. add a multiple of one row to another row ("$b \cdot R_i + R_j \rightarrow R_j$")

$$\left[\begin{array}{cc|c} 1 & * & * \\ 0 & 1 & * \end{array}\right], \text{ or } \left[\begin{array}{cc|c} 1 & * & * \\ 0 & 0 & 1 \end{array}\right] \text{ or } \left[\begin{array}{cc|c} 1 & * & * \\ 0 & 0 & 0 \end{array}\right], \ldots$$

(~~each~~ number of initial 0's in a row increases, first nonzero entry is 1)

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 2 & 1 \end{array}\right] \xrightarrow{-3R_1 + R_2 \rightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{-R_2 \rightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

Finally, we rewrite the matrix back as equations to determine the solution:

$x + y = 1$ $\quad$ $\boxed{y = 2}$ $\quad$ $x + 2 = 1$ $\quad$ $\boxed{x = -1}$ $\quad$ $\boxed{(x, y) = (-1, 2)}$

Gauss-Jordan elimination. Instead of the last step, we could have continued with one more row operation, to get a matrix in the *reduced row echelon form*:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{-R_2+R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

reduced row echelon form = echelon form (number of initial 0's in a row is increasing, first nonzero entry in each row is 1), such that above each leading 1, there are only 0's:

Ex: $\left[\begin{array}{cc|c} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 2 \end{array}\right]$, or $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 3 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$, or $\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 4 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right]$

Upshot: The solution to the system can be read off easily from the matrix:

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right] \rightsquigarrow \begin{array}{l} x\ (+0y) = -1 \\ (0x+)\ y = 2 \end{array} \to \boxed{\begin{array}{l} x = -1 \\ y = 2 \end{array}}$$

Summary (Gauss–Jordan elimination):

1. Rewrite the system of lin. equations into its associated aug. matrix.
2. Use row operations to get a matrix in row echelon form.
3. Reduce further to get the reduced row echelon form.
4. Rewrite back to equations to read off the solution.

Exercise 1. Find all solutions to the following system of equations:

$$2x + 5y + 4z = 3,$$
$$2x + 6y + 6z = 2,$$
$$3x + 10y + 11z = 2.$$

Aug. matrix:

$$\left[\begin{array}{ccc|c} 2 & 5 & 4 & 3 \\ 2 & 6 & 6 & 2 \\ 3 & 10 & 11 & 2 \end{array}\right] \xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 2 & 5 & 4 & 3 \\ 0 & 1 & 2 & -1 \\ 3 & 10 & 11 & 2 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 5/2 & 2 & 3/2 \\ 0 & 1 & 2 & -1 \\ 3 & 10 & 11 & 2 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 5/2 & 2 & 3/2 \\ 0 & 1 & 2 & -1 \\ 0 & \frac{5}{2} & 5 & -\frac{5}{2} \end{array}\right] \xrightarrow{-\frac{5}{2}R_2+R_3 \to R_3} \underbrace{\left[\begin{array}{ccc|c} 1 & 5/2 & 2 & 3/2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{row echelon form}}$$

$$\xrightarrow{-\frac{5}{2}R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \text{reduced row echelon form.}$$

Rewrite back to equations:

$$x - 3z = 4 \quad \rightsquigarrow \text{ put } z = t.$$
$$y + 2z = -1 \quad \text{Then } x = 4 + 3z = 4 + 3t,$$
$$(0 = 0) \quad y = -1 - 2z = -1 - 2t$$

$\rightsquigarrow$ solutions are $(x, y, z) = \underline{\underline{(4 + 3t, -1 - 2t, t)}}$, t any real number.

Exercise 2. Find the reduced row echelon form of the following augmented matrix:

$$\left[\begin{array}{ccc|c} 6 & 11 & 2 & 2 \\ 3 & 7 & 1 & -2 \\ 3 & 9 & 0 & -3 \end{array}\right]$$

(Corresponds to the system

$$(*)\quad \begin{aligned} 6x + 11y + 2z &= 2 \\ 3x + 7y + z &= -2 \\ 3x + 9y &= -3 \end{aligned}$$
)

$$\left[\begin{array}{ccc|c} 6 & 11 & 2 & 2 \\ 3 & 7 & 1 & -2 \\ 3 & 9 & 0 & -3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 3 & 9 & 0 & -3 \\ 3 & 7 & 1 & -2 \\ 6 & 11 & 2 & 2 \end{array}\right] \xrightarrow{-R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 3 & 9 & 0 & -3 \\ 0 & -2 & 1 & 1 \\ 6 & 11 & 2 & 2 \end{array}\right]$$

$$\xrightarrow{-2R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 3 & 9 & 0 & -3 \\ 0 & -2 & 1 & 1 \\ 0 & -7 & 2 & 8 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & -2 & 1 & 1 \\ 0 & -7 & 2 & 8 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -7 & 2 & 8 \end{array}\right] \xrightarrow{7R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{2} & 9/2 \end{array}\right] \xrightarrow{-\frac{2}{3}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{-3R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

(So the solution of (*) would be

$$(x, y, z) = (5, -2, -3)$$
)

Exercise 3. Find all solutions to the following system of equations:

$$\begin{aligned} 2x + 2y + 2z &= 5, \\ 3x + y + 5z &= 13, \\ x \qquad + 2z &= 4. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 5 \\ 3 & 1 & 5 & 13 \\ 1 & 0 & 2 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 3 & 1 & 5 & 13 \\ 2 & 2 & 2 & 5 \end{array}\right] \xrightarrow{-3R_1 + R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 1 \\ 2 & 2 & 2 & 5 \end{array}\right] \xrightarrow{-2R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & -3 \end{array}\right] \xrightarrow{-2R_2 + R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{array}\right] \xrightarrow{-\frac{1}{5}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-4R_3 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

.. corresponds to

$x + 2z = 0$

$y - z = 0$

$0 = 1$ .. inconsistent system

no solutions