Quiz 2: Lessons 1A/1B, today @ end of lecture.
Quiz 3: Lessons 2/3, Wed 2/3 @ end of lecture.

MATH 16020 Lesson 3: Substitution with Natural Log

Spring 2021

Definition. The inverse of $f(x) = e^x$ is $f^{-1}(x) = \ln(x)$, whose properties are given below:

☆ A. $\ln(e^x) = x$, $e^{\ln(x)} = x$ (e.g., $\ln(e^3) = 3$, $e^{\ln(2.6)} = 2.6$)

B. $\ln(x) = y$ + $e^y = x$ are equivalent

☆ C. Domain of $\ln(x)$ is $x > 0$, or $(0, \infty)$

D. Range of $\ln(x)$ is $(-\infty, \infty)$.

☆ E. $\ln(A^B) = B\ln(A)$

F. $\ln(AB) = \ln(A) + \ln(B)$

G. $\ln(A/B) = \ln(A) - \ln(B)$

☆ H. $\int \frac{1}{x}\,dx = \ln|x| + C$

I. $\frac{d}{dx}[\ln|x|] = \frac{1}{x}$

Example 1. Evaluate $\int_0^{\pi/20} \frac{5\sec^2(5x)}{3+\tan(5x)}\,dx$ rounded to 4 decimal places.

$$\int_0^{\pi/20} \frac{5\sec^2(5x)}{3+\tan(5x)}\,dx = \int_3^4 \frac{\cancel{5\sec^2(5x)}}{u}\,\frac{du}{\cancel{5\sec^2(5x)}} = \int_3^4 \frac{1}{u}\,du$$

$u = 3 + \tan(5x)$

$du = 5\sec^2(5x)\,dx$

$\Rightarrow dx = \frac{du}{5\sec^2(5x)}$

$x = \pi/20 \longrightarrow u = 3 + \tan(\pi/4) = 4$

$x = 0 \longrightarrow u = 3$

$$= \left[\ln|u|\right]_3^4$$

$$= \ln(4) - \ln(3)$$

$$= \ln(4/3)$$

$$\approx \boxed{0.2877}$$

Example 2. Evaluate $\int \frac{(\ln(3x^5))^2}{5x}dx = \int \frac{u^2}{5\cancel{x}} \cdot \frac{\cancel{x}}{5}du = \int \frac{u^2}{25}du = \frac{u^3}{75} + C$

$= \boxed{\frac{(\ln(3x^5))^3}{75} + C}$

$u = \ln(3x^5)$

$du = \frac{1}{3x^5} \cdot 15x^4 dx$

$du = \frac{5}{x}dx$

$\Rightarrow \boxed{dx = \frac{x}{5}du}$

Example 3. Suppose a hot air balloon is deflating in such a way that its volume changes at a rate of:

Given $V'(t) = \dfrac{2}{\sqrt[3]{t}(t^{2/3} - 25)}$ m^3/min $\quad V(0) = 15000$

min.

with $0 \leq t \leq 120$. If the volume before it starts deflating is 15000 m^3, find the volume one hour later. Round to 3 decimal places.

Find $V(60)$

$$V(60) - V(0) = \int_0^{60} \frac{2}{\sqrt[3]{t}(t^{2/3}-25)}dt = \int_{-25}^{60^{2/3}-25} \frac{\cancel{2}}{\cancel{\sqrt[3]{t}} \cdot u} \cdot \frac{3\cancel{\sqrt[3]{t}}}{\cancel{2}}du = \int_{-25}^{60^{2/3}-25} \frac{3}{u}du$$

$$= \Big[3\ln|u|\Big]_{-25}^{60^{2/3}-25}$$

$$= \boxed{3\ln|60^{2/3}-25| - 3\ln|-25|}$$

$u = t^{2/3} - 25$

$du = \frac{2}{3}\frac{1}{\sqrt[3]{t}}dt$

$\Rightarrow dt = \frac{3\sqrt[3]{t}}{2}du$

$t = 60 \longrightarrow u = 60^{2/3} - 25$
$t = 0 \longrightarrow u = -25$

$$V(60) = V(0) + (\ \ ) = 15000 + 3\ln|60^{2/3} - 25| - 3\ln(25) \approx \boxed{14997.152 \text{ m}^3}$$

Example 4. Thankfully, the person driving the hot air balloon notices the balloon deflating and so descends the balloon in a way modeled by:

$$H(t) = \frac{180}{3t - 100} \text{ meters}$$

with $60 \leq t \leq 120$ in minutes. Find the average height of the balloon over this interval. (You may have noticed that the balloon doesn't go very high in this interval of time; it's a starting business.)

Avg. of $H(t) = \dfrac{180}{3t-100}$ over $[60, 120]$.

$$H_{AVG} = \frac{\int_{60}^{120} H(t)\,dt}{120-60} = \frac{1}{60}\int_{60}^{120} \frac{180}{3t-100}\,dt = \int_{60}^{120} \frac{1}{60} \cdot \frac{180}{3t-100}\,dt$$

$$= \int_{60}^{120} \frac{3}{3t-100}\,dt$$

$u = 3t - 100$

$du = 3dt$

$\Rightarrow dt = \dfrac{du}{3}$

$\Rightarrow t = 120 \rightarrow u = 260$

$t = 60 \rightarrow u = 80$

$$= \int_{80}^{260} \frac{\cancel{3}}{u} \cdot \frac{du}{\cancel{3}}$$

$$= \int_{80}^{260} \frac{1}{u}\,du$$

$$= \Big[\ln|u|\Big]_{80}^{260}$$

$$= \ln(260) - \ln(80)$$

$$= \ln\left(\frac{260}{80}\right) = \ln\left(\frac{13}{4}\right) \approx \boxed{1.179\text{ m}}$$