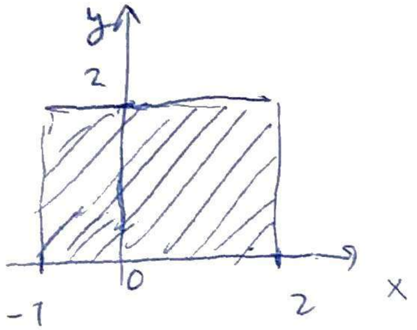
MA 16020 Lesson 29: Double integrals III

Recall (geom. interpretation of double integrals):
Given a function $z = f(x,y)$ of two variables and a region R in the xy–plane, the integral $\iint_R f(x,y)\mathrm{d}A$ has the meaning of: the volume under $z=f(x,y)$ and above the region R

Exercise 1. Compute the volume of the solid bounded by the surface $z = e^x\sqrt{y + e^x}$ from above, by the xy–plane from below and by the planes $x = -1, x = 2, y = 0$ and $y = 2$ on the sides.



$$\text{Volume} = \int_0^2\int_{-1}^2 e^x\sqrt{y+e^x}\,dx\,dy = \left|\begin{array}{l} u = e^x + y \\ du = e^x\,dx \\ x=-1 \rightsquigarrow u = e^{-1}+y \\ x = 2 \rightsquigarrow u = e^2+y \end{array}\right.$$

$$= \int_0^2\left(\int_{e^{-1}+y}^{e^2+y}\sqrt{u}\,du\right)dy = \int_0^2\left[\frac{2}{3}u^{3/2}\right]_{e^{-1}+y}^{e^2+y}dy = \int_0^2\left(\frac{2}{3}(e^2+y)^{\frac{3}{2}} - \frac{2}{3}(e^{-1}+y)^{\frac{3}{2}}\right)dy$$

$$= \underbrace{\int_0^2 \frac{2}{3}(e^2+y)^{\frac{3}{2}}\,dy}_{} - \underbrace{\int_0^2 \frac{2}{3}(e^{-1}+y)^{3/2}\,dy}_{} = \int_{e^2}^{e^2+2}\frac{2}{3}u^{\frac{3}{2}}\,du - \int_{e^{-1}}^{e^{-1}+2}\frac{2}{3}u^{\frac{3}{2}}\,du$$

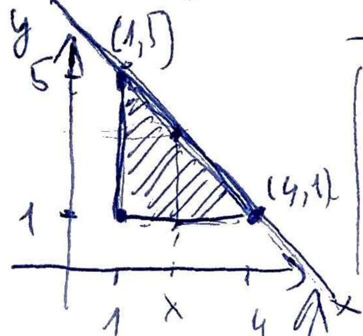
use $u = e^2+y$
$du = dy$
$y=0 \rightsquigarrow u = e^2$
$y = 2 \rightsquigarrow u = e^2+2$

use $u = e^{-1}+y$
$du = dy$
$y = 0 \rightsquigarrow u = e^{-1}$
$y = 2 \rightsquigarrow u = e^{-1}+2$

$$= \left[\frac{2}{3}\cdot\frac{2}{5}\cdot u^{\frac{5}{2}}\right]_{e^2}^{e^2+2} - \left[\frac{2}{3}\cdot\frac{2}{5}\cdot u^{5/2}\right]_{e^{-1}}^{e^{-1}+2} = \frac{4}{15}\left((e^2+2)^{\frac{5}{2}} - e^5 - (e^{-1}+2)^{\frac{5}{2}} + e^{-\frac{5}{2}}\right)$$

$$(\approx 30.176)$$

Exercise 2. Compute the volume under the surface $z = x^2y$ and above the triangle with vertices $(1,1,0)$, $(1,5,0)$ and $(4,1,0)$.



$$1 \le x \le 4$$
$$1 \le y \le -\frac{4}{3}x + \frac{19}{3}$$

$$y = ax + b = -\frac{4}{3}x + \frac{19}{3}$$

$$\boxed{\begin{aligned} 5 &= a\cdot 1 + b \\ 1 &= a\cdot 4 + b \end{aligned}}$$

$$a = 5 - b$$
$$1 = (5-b)\cdot 4 + b$$
$$1 = 20 - 4b + b$$
$$1 = 20 - 3b$$
$$3b = 19$$
$$\boxed{b = \frac{19}{3}} \qquad a = 5 - \frac{19}{3} = \frac{15-19}{3} = -\frac{4}{3}$$

$$\text{Volume} = \int_1^4 \int_1^{\frac{19}{3} - \frac{4}{3}x} x^2 y \, dy \, dx =$$

$$= \int_1^4 \left[\frac{x^2 y^2}{2} \right]_1^{\frac{19}{3} - \frac{4}{3}x} dx =$$

$$= \int_1^4 \left(\frac{x^2 \left(\frac{19}{3} - \frac{4}{3}x\right)^2}{2} - \frac{x^2 \cdot 1}{2} \right) dx$$

$$= \int_1^4 \left(\frac{x^2}{2} \left(\left(\frac{19}{3}\right)^2 - 2\cdot\left(\frac{19}{3}\right)\cdot\left(\frac{4}{3}\right)x + \left(\frac{4}{3}\right)^2 x^2 \right) - \frac{x^2}{2} \right) dx$$

$$= \int_1^4 \left(\frac{361}{18}x^2 - \frac{1}{2}x^2 - \frac{152}{18}x^3 + \frac{16}{18}x^4 \right) dx$$

$$= \left[\frac{352}{18}\frac{x^3}{3} - \frac{152}{18}\frac{x^4}{4} + \frac{16}{18}\frac{x^5}{5} \right]_1^4$$

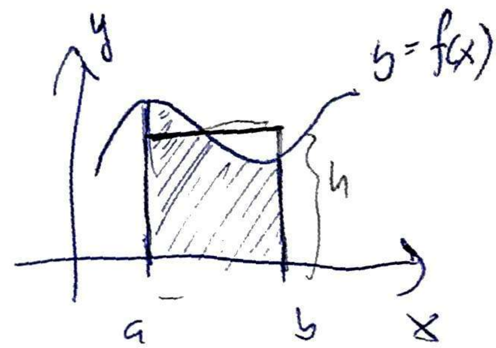
$$= \left(\frac{352}{18}\frac{4^3}{3} - \frac{152}{18}\cdot\frac{4^4}{4} + \frac{16}{18}\cdot\frac{4^5}{5} \right) - \left(\frac{352}{18}\cdot\frac{1}{3} - \frac{152}{18}\cdot\frac{1}{4} + \frac{16}{18}\cdot\frac{1}{5} \right)$$

$$= \underline{54.2}$$

Recall;

If ~~f~~ $y=f(x)$ is a function of one variable, the average of f over the interval $[a,b]$ is given by

$$f_{avg} = \frac{1}{b-a} \underbrace{\int_a^b f(x)\,dx}_{\text{Area under } f}$$



If $z=f(x,y)$ is a function of two variables, and R is a region ~~then~~ in the x,y-plane, then the average value of f over the region R is given as

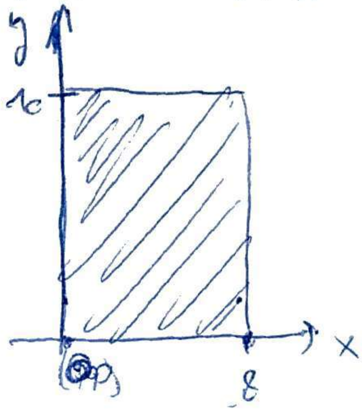
A = area of R

$$\boxed{f_{avg} = \frac{1}{A} \iint_R f(x,y)\,dA},$$

Exercise 3. There is a heater in a corner of a rectangular room of dimensions 8×10 m. As a result, the temperature in $^\circ C$ in the room is described by

$$T(x, y) = 60 - 0.3(x^2 + y^2),$$

where (x, y) are the coordinates of a given point in the room (the heater is placed at $(0, 0)$). What is the average temperature in the room?



$$T_{avg} = \frac{1}{80}\int_0^8\int_0^{10} \left(60 - 0.3(x^2+y^2)\right) dy\, dx$$

$$= \frac{1}{80}\int_0^8 \left[60y - 0.3x^2y - 0.3\frac{y^3}{3}\right]_0^{10} dx =$$

Area = 8·10 = 80

$$= \frac{1}{80}\int_0^8 (600 - 3x^2 - 100)\, dx =$$

$$= \frac{1}{80}\int_0^8 (500 - 3x^2)\, dx = \frac{1}{80}\left[500x - x^3\right]_0^8 =$$

$$= \frac{1}{80}(500 \cdot 8 - 8^3) = 50 - \frac{64}{10} = \frac{436}{10} =$$

$$= 43.6$$

Exercise 4. The water temperature in a lake during the night is given approximately (in $°F$) by the function

$$T(d,t) = \frac{350e^{-0.05t}}{d+5}$$

where t is the number of hours that passed since 8 pm and d is the depth in m. What is the average temperature of the water from the surface to the depth of 5 m and between 10 pm and 1 am?

Want: T_{avg} for $0 \le d \le 5$,
$2 \le t \le 5$

$$\rightsquigarrow T_{avg} = \frac{1}{5\cdot(5-2)} \cdot \int_0^5 \int_2^5 \frac{350e^{-0.05t}}{d+5}\, dt\, dd =$$

$$= \frac{1}{15} \cdot \int_0^5 \left[\frac{350}{d+5} \cdot \left(\frac{1}{-0.05}\right) \cdot e^{-0.05t} \right]_2^5 dd$$

$$= \frac{1}{15} \int_0^5 \frac{-7000}{d+5} \cdot \left(e^{-0.25} - e^{-0.1}\right) dd = \left| \begin{matrix} u = d+5 \\ du = dd \end{matrix} \right|$$

$$= \frac{1}{15} \int_5^{10} \frac{-7000\left(e^{-0.25} - e^{-0.1}\right)}{u}\, du = \cancel{\frac{1}{15}\left[-7000\left(e^{-0.25} - e^{-0.1}\right) \ln(u)\right]}$$

$$= \frac{1}{15} \left[7000\left(e^{-0.1} - e^{-0.25}\right) \cdot \ln(u) \right]_5^{10}$$

$$= \underline{\underline{\frac{7000}{15} \cdot \left(e^{-0.1} - e^{-0.25}\right) \cdot \left(\ln(10) - \ln(5)\right)}} \quad (\approx 40.7689)$$