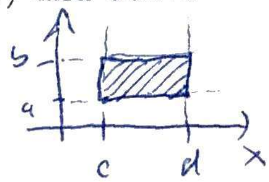
MA 16020 Lesson 28: Double integrals II

Recall (integrals over rectangles): Given a function $z = f(x, y)$ of two variables, the double integrals

$$\int_a^b\left(\int_c^d f(x,y)\mathrm{d}x\right)\mathrm{d}y, \quad \int_c^d\left(\int_a^b f(x,y)\mathrm{d}y\right)\mathrm{d}x$$

both compute the volume of the region below $z = f(x, y)$ and above the rectangle:

$R = \{(x,y) \mid c \le x \le d,\ a \le y \le b\}$

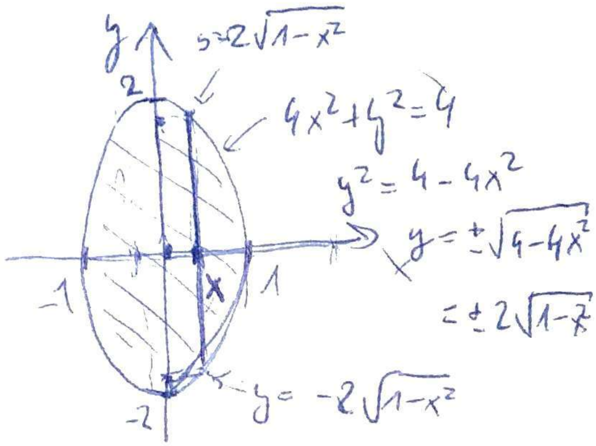


Let us denote this common integral by $\underline{\iint_R f(x,y)\mathrm{d}A}$.

Integrals over regions in a plane. Given a (reasonably complicated/simple) region R in the xy–plane, the integral $\iint_R f(x,y)\mathrm{d}A$ still makes sense. To compute it, we may consider two basic ways of going through all of the points of the region:

Example. Let R be the the interior of the ellipse $4x^2 + y^2 = 4$. We want to compute $\iint_R(\sqrt{4-y^2} + 2x)\mathrm{d}A$.

1. **First way:** Give lower and upper limits for the x-coordinates of points present, then give lower and upper limits (posssibly depending on x) for the corresponding y-coordinates.



$-1 \le x \le 1$

for fixed x,

$-2\sqrt{1-x^2} \le y \le 2\sqrt{1-x^2}$

This way, we obtain

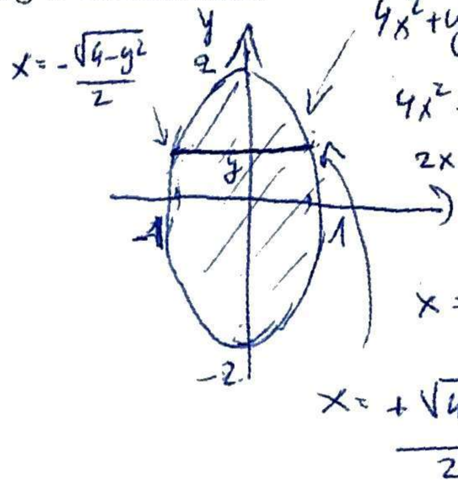
$$\iint_R(\sqrt{4-y^2} + 2x)\mathrm{d}A = \underbrace{\int_{-1}^{1}\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}\left(\sqrt{4-y^2}+2x\right)dy\,dx}_{\text{complicated}}$$

2. **Second way (the other way round):** Give lower and upper limits for the y-coordinates of points present, then give lower and upper limits (posssibly depending on y) for the corresponding x-coordinates.

$$-2 \le y \le 2$$

Depending on y,

$$-\frac{\sqrt{4-y^2}}{2} \le x \le \frac{\sqrt{4-y^2}}{2}$$



$4x^2+y^2=4$

$4x^2 = 4-y^2$

$2x = \pm\sqrt{4-y^2}$

$x = \pm\frac{\sqrt{4-y^2}}{2}$

This way, we obtain

$$\iint_R (\sqrt{4-y^2}+2x)\mathrm{d}A = \int_{-2}^{2}\int_{-\frac{\sqrt{4-y^2}}{2}}^{\frac{\sqrt{4-y^2}}{2}} \left(\sqrt{4-y^2}+2x\right)dx\,dy$$

Finally, let us compute the integral:

$$\int_{-2}^{2}\left(\int_{-\frac{\sqrt{4-y^2}}{2}}^{\frac{\sqrt{4-y^2}}{2}} \left(\sqrt{4-y^2}+2x\right)dx\right)dy = \int_{-2}^{2}\left[\sqrt{4-y^2}\cdot x + x^2\right]_{-\frac{\sqrt{4-y^2}}{2}}^{\frac{\sqrt{4-y^2}}{2}} dy =$$

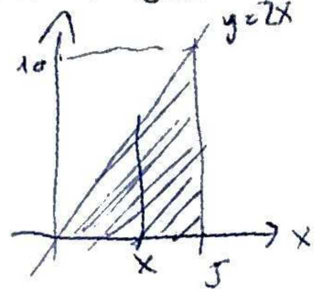
$$= \int_{-2}^{2}\left(\left(\sqrt{4-y^2}\cdot\frac{\sqrt{4-y^2}}{2} + \left(\frac{\sqrt{4-y^2}}{2}\right)^2\right) - \left(\sqrt{4-y^2}\left(-\frac{\sqrt{4-y^2}}{2}\right) + \left(-\frac{\sqrt{4-y^2}}{2}\right)^2\right)\right)dy$$

$$= \int_{-2}^{2}\left(\left(\frac{4-y^2}{2} + \frac{4-y^2}{4}\right) - \left(-\frac{4-y^2}{2} + \frac{4-y^2}{4}\right)\right)dy$$

$$= \int_{-2}^{2}(4-y^2)\,dy = \left[4y - \frac{y^3}{3}\right]_{-2}^{2} = \left(4\cdot 2 - \frac{2^3}{3}\right) - \left(4\cdot(-2) - \frac{(-2)^3}{3}\right)$$

$$= 16 - \frac{16}{3} = \frac{32}{3}$$

Exercise 1. Evaluate the integral $\iint_R (x^2+y^2)\mathrm{d}A$, where R is the region bounded by the lines $y=2x$, $x=5$ and the x-axis.



$0 \le x \le 5$
$0 \le y \le 2x$

$$\iint_R (x^2+y^2)\,dA = \int_0^5 \int_0^{2x} (x^2+y^2)\,dy\,dx =$$

$$= \int_0^5 \left[x^2 y + \frac{y^3}{3}\right]_0^{2x} dx = \int_0^5 \left(x^2\cdot(2x) + \frac{(2x)^3}{3}\right) dx =$$

$$= \int_0^5 \frac{14}{3}x^3\,dx = \left[\frac{14}{12}x^4\right]_0^5 =$$

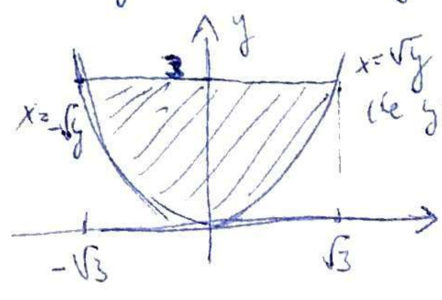
$$= \frac{14\cdot 5^4}{12} \quad (\approx 729.167)$$

Exercise 2. Switch the order of integration for the integral

$$(*) = \int_0^3 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\mathrm{d}x\mathrm{d}y\,.$$

Need to sketch the region indicated by integration limits:

$0 \le y \le 3$,
$-\sqrt{y} \le x \le \sqrt{y}$



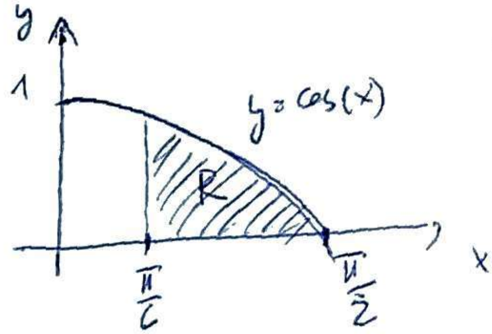
⇝ get $-\sqrt{3} \le x \le \sqrt{3}$,
$x^2 \le y \le 3$

$$\rightsquigarrow (*) = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} f(x,y)\,dy\,dx$$

Exercise 3. Evaluate the integral

$$\iint_R 6\sin^2(x)\mathrm{d}A\,,$$

where R is the region bouded by the curves $y = \cos(x)$, $x = \pi/6, x = \pi/2$ and the x–axis.



$$\rightsquigarrow \frac{\pi}{6} \le x \le \frac{\pi}{2},$$
$$0 \le y \le \cos(x)$$

$$\rightsquigarrow \iint_R 6\sin^2(x)\,dA = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(\int_0^{\cos(x)} 6\sin^2(x)\,dy\right)dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\Big[6\sin^2(x)\,y\Big]_0^{\cos(x)}dx =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 6\sin^2(x)\cos(x)\,dx = \left|\begin{array}{l} u = \sin(x) \\ du = \cos(x)\,dx \\ x = \frac{\pi}{6} \rightsquigarrow u = \frac{1}{2} \\ x = \frac{\pi}{2} \rightsquigarrow u = 1 \end{array}\right| = \int_{\frac{1}{2}}^{1} 6u^2\,du =$$

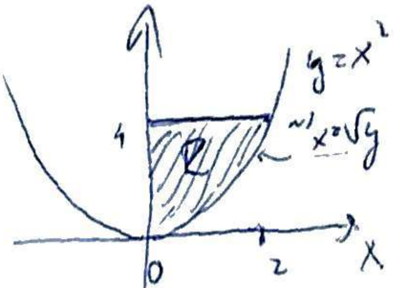
$$= \Big[2u^3\Big]_{\frac{1}{2}}^{1} = 2 - 2\cdot\left(\frac{1}{2}\right)^3 = 2 - \frac{1}{4} = \underline{\underline{\frac{7}{4}}}$$

Exercise 4 (time permitting). Evaluate the integral

$$(*) = \int_0^2 \int_{x^2}^4 2x\sqrt{3+y^2}\,\mathrm{d}y\mathrm{d}x\ ,$$

directly .. difficult (integrating $\sqrt{3-y^2}$) ⇝ we switch order of integration:

$$0 \le x \le 2,\quad x^2 \le y \le 4 \quad ⇝$$



$$⇝ \boxed{0 \le y \le 4,\quad 0 \le x \le \sqrt{y}}$$

$$(*) = \int_0^4 \left(\int_0^{\sqrt{y}} 2x\sqrt{3+y^2}\,dx \right) dy = \int_0^4 \left[x^2 \cdot \sqrt{3+y^2} \right]_0^{\sqrt{y}} dy =$$

$$= \int_0^4 y \cdot \sqrt{3+y^2}\,dy = \left| \begin{array}{l} u = 3+y^2 \\ du = +2y\,dy \\ y=0 ⇝ u=3 \\ y=4 ⇝ u=19 \end{array} \right| = \int_3^{19} \frac{\sqrt{u}}{(2)}\,du =$$

$$= \left[\frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} \right]_3^{19} = \left[\frac{1}{3} u^{\frac{3}{2}} \right]_3^{19} = \frac{1}{3}\left(19^{\frac{3}{2}} - 3^{\frac{3}{2}} \right)$$

$$(\approx 25.874)$$