MA 16020 Lesson 27: Double integrals I

Let $z = f(x, y)$ be a function of two variables. Similarly to taking (partial) derivatives with respect to x and y, we can take

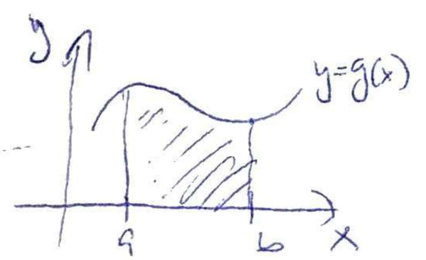
1. $\int f(x, y)\mathrm{d}x \ldots$ "integrating with respect to x" – y is treated as a constant
2. $\int f(x, y)\mathrm{d}y \ldots$ "integrating with respect to y" – x is treated as a constant

We will be mostly concerned with definite integrals. Combining the above two ways of integration, we obtain a double integral:

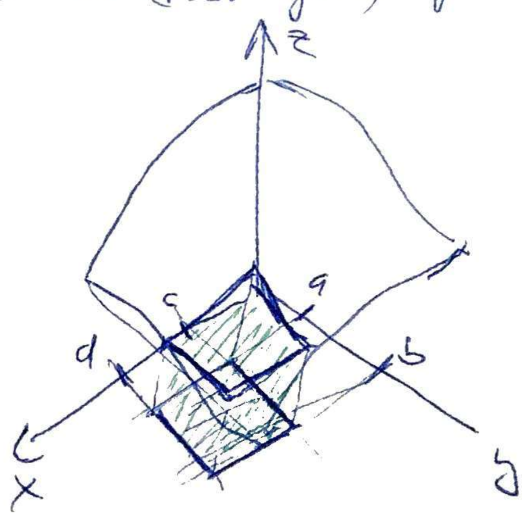
$$\int_a^b \int_c^d f(x, y)\mathrm{d}x\mathrm{d}y = \int_a^b \left(\int_c^d f(x,y)\,dx \right) dy$$

$$\int_a^b \int_c^d f(x, y)\mathrm{d}y\mathrm{d}x = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Recall that for a function $g(x)$ of one variable, the meaning of the integral $\int_a^b g(x)\mathrm{d}x$ is: the area between the curve $y=g(x)$ and the x-axis, over the interval $[a, b]$



Similarly, the geometric meaning of the double integral $\int_a^b \int_c^d f(x, y)\mathrm{d}x\mathrm{d}y$ is: the volume of the region below $z = f(x,y)$ and above the xy-plane, over the (rectangular) region $\{(x,y) \mid c \le x \le d,\ a \le y \le b\}$



Exercise 1. Evaluate the integral

$$\int_2^4 \int_0^5 \frac{x^3}{y^2} \mathrm{d}x\mathrm{d}y \,.$$

$$\int_2^4 \left(\int_0^5 \frac{x^3}{y^2}\,dx\right)dy = \int_2^4 \left[\frac{x^4}{4}\cdot\frac{1}{y^2}\right]_0^5 dy = \int_2^4 \left(\frac{5^4}{4}\cdot\frac{1}{y^2} - \frac{0^4}{4}\cdot\frac{1}{y^2}\right)dy$$

$$= \int_2^4 \frac{5^4}{4}\cdot y^{-2}\,dy = \left[\frac{5^4}{4}\,\frac{y^{-1}}{(-1)}\right]_2^4 = -\frac{5^4}{4}\cdot 4^{-1} - \left(-\frac{5^4}{4}\cdot 2^{-1}\right)$$

$$= -\frac{625}{16} + \frac{625}{8} = \underline{\underline{\frac{625}{16}}} \;(\approx 39.0625)$$

Exercise 2. Evaluate the integral

$$\int_{-1}^{2} \int_0^{\pi/3} 2y^3 \sin(x)\mathrm{d}x\mathrm{d}y \,.$$

$$\int_{-1}^{2}\left(\int_0^{\frac{\pi}{3}} 2y^3 \sin(x)\,dx\right)dy = \int_{-1}^{2}\left[2y^3\cdot(-\cos(x))\right]_0^{\frac{\pi}{3}} dy =$$

$$= \int_{-1}^{2}\left(-2y^3\cdot\cos\left(\frac{\pi}{3}\right) + 2y^3\cdot\cos(0)\right)dy = \int_{-1}^{2}\left(-2y^3\,\frac{1}{2} + 2y^3\cdot 1\right)dy$$

$$= \int_{-1}^{2} 2y^3\left(-\frac{1}{2}+1\right)dy = \int_{-1}^{2} y^3\,dy = \left[\frac{y^4}{4}\right]_{-1}^{2} =$$

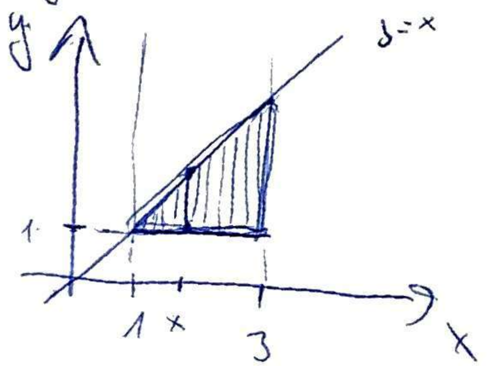
$$= \frac{2^4}{4} - \frac{(-1)^4}{4} = 4 - \frac{1}{4} = \underline{\underline{\frac{15}{4}}} \;(= 3.75)$$

Sometimes the limits of the "inner integral" may depend on the unused variable. This corresponds to integration over regions that are not necessarily rectangular.

Example.

$$\int_1^3 \int_1^x x^2 y \, dy \, dx$$

$\rightsquigarrow$ we are integrating the function $f(x,y) = x^2 y$ over the region $\{ (x,y) \mid 1 \le x \le 3, \ 1 \le y \le x \}$



$$\int_1^3 \left(\int_1^x x^2 y \, dy \right) dx = \int_1^3 \left[\frac{x^2 y^2}{2} \right]_1^x dx = \int_1^3 \left(\frac{x^2 \cdot x^2}{2} - \frac{x^2 \cdot 1^2}{2} \right) dx$$

$$= \int_1^3 \left(\frac{x^4}{2} - \frac{x^2}{2} \right) dx = \left[\frac{x^5}{5 \cdot 2} - \frac{x^3}{3 \cdot 2} \right]_1^3 =$$

$$= \left(\frac{3^5}{10} - \frac{3^3}{6} \right) - \left(\frac{1^5}{10} - \frac{1^3}{6} \right) = \frac{243}{10} - \frac{27}{6} - \frac{1}{10} + \frac{1}{6} =$$

$$= \frac{242}{10} - \frac{26}{6} = \frac{298}{15} \quad (\approx 19.87)$$

Exercise 3. Evaluate the integral

$$\int_0^{\sqrt{\pi/6}} \int_{-y^2}^{0} y\cos(x) \mathrm{d}x \mathrm{d}y \, .$$

$$\int_0^{\sqrt{\pi/6}} \left(\int_{-y^2}^{0} y \cos x \, dx \right) dy = \int_0^{\sqrt{\frac{\pi}{6}}} \Big[y \sin x \Big]_{-y^2}^{0} dy =$$

$$= \int_0^{\sqrt{\frac{\pi}{6}}} \Big(\underbrace{y \cdot \sin(0)}_{=0} - y \cdot \underbrace{\sin(-y^2)}_{=-\sin(y^2)} \Big) dy = \int_0^{\sqrt{\frac{\pi}{6}}} y \sin(y^2) \, dy =$$

$$= \left| \begin{array}{l} u = y^2 \\ du = 2y \, dy \\ y = 0 \rightsquigarrow u = 0 \\ y = \sqrt{\frac{\pi}{6}} \rightsquigarrow u = \left(\sqrt{\frac{\pi}{6}}\right)^2 = \frac{\pi}{6} \end{array} \right| = \frac{1}{2} \int_0^{\frac{\pi}{6}} \sin(u) \, du = \frac{1}{2} \Big[-\cos u \Big]_0^{\frac{\pi}{6}} =$$

$$= \frac{1}{2} \left(-\frac{\sqrt{3}}{2} + \cancel{\cos 0} \, 1 \right) = \underline{\underline{\frac{2-\sqrt{3}}{4}}} \; (\approx 0.067)$$

(it is an integration of $f(x,y) = y\cos(x)$ over the region

$$\left\{ (x,y) \;\middle|\; 0 \leq y \leq \sqrt{\frac{\pi}{6}}, \; -y^2 \leq x \leq 0 \right\}$$