MA 16020 Lesson 26: Lagrange multipliers II

Recall (constrained min/max using Lagrange multipliers):
When trying to minimize/maximize the value of the function $z = f(x, y)$ subject to the constraint $g(x, y) = C$, the critical points are given as points (x, y) that are solutions to the system of the equations:

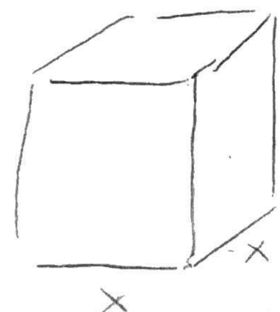
(*) $f_x = \lambda \cdot g_x$, $f_y = \lambda \cdot g_y$ (λ - new variable "Lagrange multiplier")

as well as the original constraint:

$$g(x,y) = C$$

typical strategy to solve the system (not always): Solve (*) for λ, then make the RHS's equal, then use $g(x,y) = C$

Exercise 1. The prices for constructing a box with a sqaure base are: \$15 per m^2 for the top, \$12 per m^2 for the bottom, and \$6 per m^2 for the sides. What is the biggest possible volume of the box for \$100?



maximize volume $= x^2 y$

subject to the constraint cost $= 15x^2 + 12x^2 + 6 \cdot 4 \cdot xy$

$= 27x^2 + 24xy = 100$

$$\boxed{\begin{aligned} 2xy &= \lambda \cdot (54x + 24y) \\ x^2 &= \lambda \cdot (24x) \\ 27x^2 + 24xy &= 100 \end{aligned}}$$

$\leadsto \lambda = \dfrac{2xy}{(54x+24y)}$ (or $54x + 24y = 0$) ← may disregard (cannot be satisfied with $x, y > 0$)

$\leadsto \lambda = \dfrac{x^2}{24x} = \dfrac{x}{24}$ (or $x = 0$)

$\leadsto \dfrac{2xy}{54x+24y} = \dfrac{x}{24}$ $/ : x$

$\dfrac{2y}{54x+24y} = \dfrac{1}{24}$

$48y = 54x + 24y$

$24y = 54x$, $\underline{y = \dfrac{54}{24} x = \dfrac{9}{4} x}$

$27x^2 + 24 \cdot x \cdot \left(\dfrac{9}{4} x\right) = 100$

$27x^2 + 54x^2 = 100$

$81x^2 = 100$

$x^2 = \dfrac{100}{81}$

$x = \pm\sqrt{\dfrac{100}{81}} = \pm\dfrac{10}{9}$

$x > 0 \leadsto \underline{x = \dfrac{10}{9}}$, $\underline{y = \dfrac{9}{4} \cdot \dfrac{10}{9} = \dfrac{10}{4} = \dfrac{5}{2}}$

Optimal dimensions:

$\dfrac{10}{9} \times \dfrac{10}{9} \times \dfrac{5}{2}$ meters,

Maximal volume

$= \left(\dfrac{10}{9}\right)^2 \cdot \dfrac{5}{2} = \dfrac{500}{162}\ m^3$

Exercise 2. Given that a company spends x thousands of dollars on internet advertising and y thousands of dollars on other forms of advertising, it is expected to sell

$$S(x,y) = 15000 + 100x^{1.5}y^{0.5}$$

units of their product. How should the company distribute its advertising budget of \$200 000 to achieve maximum sales?

Maximize $S(x,y) = 15000 + 100x^{1.5}y^{0.5}$

subject to $x+y=200$

$$\boxed{\begin{aligned} 100\cdot(1.5)x^{0.5}\cdot y^{0.5} &= \lambda\cdot 1 \\ 100\cdot(0.5)\,x^{1.5}y^{-0.5} &= \lambda\cdot 1 \\ x+y &= 200 \end{aligned}}$$

$150\,x^{0.5}\cdot y^{0.5} = 50\,x^{1.5}y^{-0.5} \;/\cdot y^{0.5}\cdot x^{-0.5}$

$150\,y^{1} = 50\,x^{1} \;/:50$

$3y = x$ (or $x=0$ or $y=0$)

(A) $x=0$

$0+y=200$ $y=200$

crit pt $(0,200)$

(B) $y=0$:

$x+0=200$

$x=200$

Crit pt $(200,0)$

(C) $3y=x$

$3y+y=200$

$4y=200$

$y=50$ $x=3\cdot 50=150$

Evaluate S at crit. pts:

$S(0,200) = 15000 + 100\cdot 0$
$= 15000,$

$S(200,0) = 15000 + 100\cdot 0$
$= 15000$

$S(150,50) = 15000 + 100\cdot(150)^{1.5}\cdot(50)^{0.5}$

$\approx 1\,314\,038$

maximum

optimal distribution is

\$150 000 to internet ads,

\$50 000 to other ads

Exercise 3. In a certain region, the population of rabbits R (in hundreds of specimens) and the population of foxes F (in hundreds of specimens) satisfy the equation

$$2(R-15)^2 + 6(F-10)^2 = 144.$$

What is the maximal and minimal possible combined number of rabbits and foxes living in the region?

Maximize/minimize $R+F$

Subject to constraint $2(R-15)^2+6(F-10)^2=144$

$$\leadsto \begin{cases} 1 = \lambda \cdot 4(R-15) \\ 1 = \lambda \cdot 12(F-10) \\ 2(R-15)^2+6(F-10)^2=144 \end{cases}$$

$\leadsto \lambda = \frac{1}{4(R-15)}$ (or $R-15=0$) $\leftarrow$ may disregard

$\leadsto \lambda = \frac{1}{12(F-10)}$ (or $F-10=0$) $\leftarrow$ (the other don't provide solutions)

$$\frac{1}{4(R-15)} = \frac{1}{12(F-10)}$$

$$12(F-10) = 4(R-15)$$

$$12F - 120 = 4R - 60$$

$$4R = 12F - 60$$

$$R = 3F - 15$$

$$2(3F-15-15)^2 + 6(F-10)^2 = 144$$

$$2(3F-30)^2 + 6(F-10)^2 = 144$$

$$2(3(F-10))^2 + 6(F-10)^2 = 144$$

$$2\cdot 9\cdot (F-10)^2 + 6(F-10)^2 = 144$$

$$24(F-10)^2 = 144$$

$$(F-10)^2 = \frac{144}{24} = 6$$

$$F - 10 = \pm\sqrt{6}$$

$$F = 10 \pm \sqrt{6}$$

$F = 10+\sqrt{6} \leadsto R = 3(10+\sqrt{6})-15 = 15+3\sqrt{6}$

$F = 10-\sqrt{6} \leadsto R = 3(10-\sqrt{6})-15 = 15-3\sqrt{6}$

Population$(15+3\sqrt{6}, 10+\sqrt{6}) = 25+4\sqrt{6} \approx 34.8$ hundreds of specimens $\boxed{\text{MAX}}$

Population$(15-3\sqrt{6}, 10-\sqrt{6}) = 25-4\sqrt{6} \approx 15.2$ hundreds of specimens $\boxed{\text{MIN}}$

Exercise 4. If a certain strain of bacteria is fed by x grams of nutrient A, y grams of nutrient B, it will ultimately produce $x^{0.6}y^{0.4}$ grams of a desired chemical. The cost of the nutrients are: 15 dollars per gram for nutrient A and 11 dollars per gram of nutrient B. What is the minimal cost to produce 50 grams of the desired chemical?

Want to minimize cost $= 15x + 11y$ subject to constraint $x^{0.6}y^{0.4} = 50$

$$\rightsquigarrow \boxed{\begin{aligned} 15 &= \lambda \cdot (0.6) \cdot x^{-0.4} y^{0.4} \\ 11 &= \lambda \cdot (0.4) \cdot x^{0.6} \cdot y^{-0.6} \\ x^{0.6} y^{0.4} &= 50 \end{aligned}}$$

$$\rightsquigarrow \lambda = \frac{15}{0.6} x^{0.4} \cdot y^{-0.4}$$

$$\rightsquigarrow \lambda = \frac{11}{0.4} x^{-0.6} y^{0.6}$$

($x=0$, $y=0$ we may disregard)

$$\frac{15}{0.6} x^{0.4} y^{-0.4} = \frac{11}{0.4} x^{-0.6} y^{0.6}$$

$$\frac{15}{0.6} x^{1} = \frac{11}{0.4} y^{1}$$

$$60x = 66y$$

$$x = \frac{11}{10} y$$

plug into $x^{0.6}y^{0.4} = 50$

$$\left(\frac{11}{10}y\right)^{0.6} y^{0.4} = 50$$

$$\left(\frac{11}{10}\right)^{0.6} y^{0.6} \cdot y^{0.4} = 50$$

$$\left(\frac{11}{10}\right)^{0.6} \cdot y = 50$$

$$y = 50 \Big/ \left(\frac{11}{10}\right)^{0.6} \approx 47.22$$

$$x = \frac{11}{10} y \approx 51.94$$

$$\left(= \left(\frac{11}{10}\right)^{0.4} \cdot 50\right)$$

The only critical point to consider is $\left(\left(\frac{11}{10}\right)^{0.4} \cdot 50, \left(\frac{11}{10}\right)^{-0.6} \cdot 50\right)$, in which case the cost is

$$15 \cdot \left(\left(\frac{11}{10}\right)^{0.4} \cdot 50\right) + 11 \cdot \left(\frac{11}{10}\right)^{-0.6} \cdot 50$$

≈ 1298.58 dollars