MA 16020 Lesson 23: Extrema of functions of two variables I

Recall (derivative tests for local max, min of a function):
Let $y = f(x)$ be a function of one variable.

1. Its local maxima and minima are among the points x_0 which satisfy
   $f'(x_0) = 0$ (= x_0 is critical pt) *(first derivative test).*

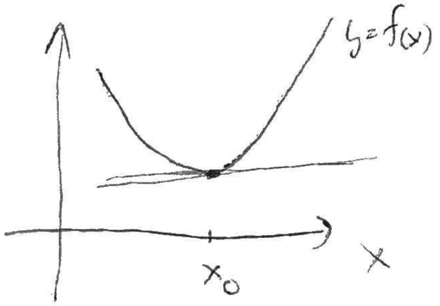
2. Given such a point x_0, to determine whether x_0 is a point of local maximum or minimum, we look at $f''(x_0)$:

   If $f''(x_0) > 0$, then x_0 is a local minimum of f.

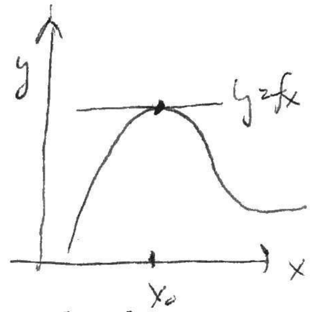
   If $f''(x_0) < 0$, then x_0 is a local maximum of f.

   If $f''(x_0) = 0$, then the test is inconclusive for this x_0.

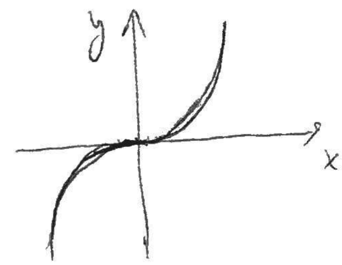
   (second derivative test).



local min.



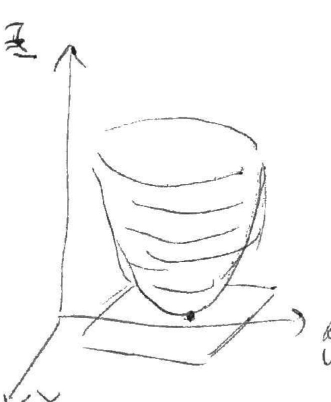
local max.



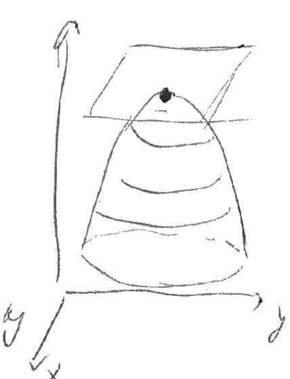
neither

Maxima and minima in two variables:

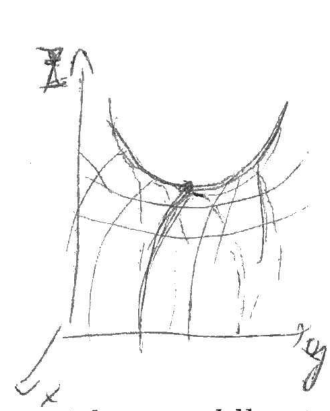
If $z = f(x, y)$ is a function of two variables, the folowing extrema may occur:



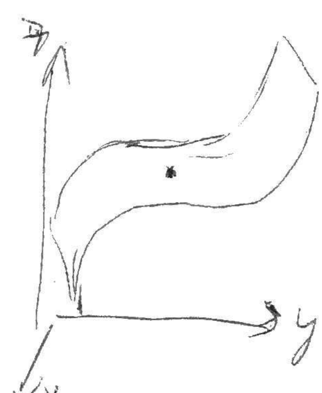
local min.



local max.



neither - saddle pt.



neither - "other"

We see that in all the cases of extrema, the tangent plane to the graph is parallel to the xy plane (horizontal), which can be described in terms of first partial derivatives as:

$$\boxed{\frac{\partial f}{\partial x}(x_0,y_0)=0 \quad \& \quad \frac{\partial f}{\partial y}(x_0,y_0)=0}$$

(first derivative test)

To determine what type of extreme (if any) is taking place, we use an analogous second derivative test. To perform it, we compute the *discriminant* at the given critical point:

$$D = D(x_0, y_0) = f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)\cdot f_{yx}(x_0,y_0) = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$$

If $D>0$ & $f_{xx}(x_0,y_0)>0$, then (x_0, y_0) is a local minimum of f.

If $D>0$ & $f_{xx}(x_0,y_0)<0$, then (x_0, y_0) is a local maximum of f.

If $D<0$, then (x_0, y_0) is a saddle point of f.

If $D=0$, then the test is inconclusive for this (x_0, y_0).

Summary (Finding extrema of functions of two variables).

1. Find all the *critical points:* Points (x, y) satisfying:

   $$\frac{\partial f}{\partial x}(x,y)=0, \quad \frac{\partial f}{\partial y}(x,y)=0$$

2. Compute all the second-order partial derivatives of f and $D = f_{xx}\cdot f_{yy} - f_{xy}^2$

3. For a given critical point (x_0, y_0), evaluate D and f_{xx} at (x_0, y_0).

   If $D>0$ & $f_{xx}(x_0,y_0)>0$, then (x_0, y_0) is a local minimum of f.

   If $D>0$ & $f_{xx}(x_0,y_0)<0$, then (x_0, y_0) is a local maximum of f.

   If $D<0$, then (x_0, y_0) is a saddle point of f.

   If $D=0$, then the test is inconclusive for this (x_0, y_0).

Exercise 1. Find all the local maxima, minima and saddle points of the function
$$f(x,y) = x^3 - \frac{2}{3}y^3 - 2y^2 - 36x + 6y.$$

1. Solve $\frac{\partial f}{\partial x} = 0$, & $\frac{\partial f}{\partial y} = 0$ " (critical points)

$\frac{\partial f}{\partial x} = 3x^2 - 36$, $\frac{\partial f}{\partial y} = -2y^2 - 4y + 6$

$3x^2 - 36 = 0$ & $-2y^2 - 4y + 6 = 0$ $/ :(-2)$

$x^2 - 12 = 0$ $y^2 + 2y - 3 = 0$

$x^2 = 12$

$x = \pm\sqrt{12} = \pm 2\sqrt{3}$

$y_{1,2} = \frac{-2 \pm \sqrt{4+4\cdot 3}}{2} = \frac{-2 \pm \sqrt{16}}{2} = \begin{cases} 1 \\ -3 \end{cases}$

→ critical points are $(2\sqrt{3}, 1)$, $(-2\sqrt{3}, 1)$, $(2\sqrt{3}, -3)$, $(-2\sqrt{3}, -3)$.

2. compute D, f_{xx}:

$f_{xx} = \frac{\partial}{\partial x}(3x^2 - 36) = 6x$, $f_{xy} = \frac{\partial}{\partial y}(3x^2 - 36) = 0$

$f_{yy} = \frac{\partial}{\partial y}(-2y^2 - 4y + 6) = -4y - 4$ $D = 6x(-4y-4) = -24x(y+1)$

3. Test the critical points

- $(2\sqrt{3}, 1)$: $f_{xx} = 6\cdot 2\sqrt{3} > 0$
  $D = -24\cdot 2\sqrt{3}\cdot 2 < 0$
  saddle pt
- $(2\sqrt{3}, -3)$: $f_{xx} = 6\cdot 2\sqrt{3} > 0$,
  $D = -24\cdot 2\sqrt{3}\cdot(-2) = 96\sqrt{3} > 0$
  local min
- $(-2\sqrt{3}, 1)$: $f_{xx} = -6\cdot 2\sqrt{3} < 0$
  $D = -24\cdot(-2\sqrt{3})\cdot 2 = 96\sqrt{3} > 0$
  local max
- $(-2\sqrt{3}, -3)$: $f_{xx} = 6\cdot(-2\sqrt{3}) < 0$
  $D = (-24)\cdot(-2\sqrt{3})\cdot(-2) = -96\sqrt{3} < 0$
  saddle point

Exercise 2. Find all the local maxima, minima and saddle points of the function

$$f(x,y) = \frac{2}{3}y^3 + x^2 - 4yx - 10y + 6.$$

1. critical points:

$$\frac{\partial f}{\partial x} = 2x - 4y, \quad \frac{\partial f}{\partial y} = 2y^2 - 4x - 10$$

$\rightsquigarrow$ $2x - 4y = 0$ & $2y^2 - 4x - 10 = 0$

$x = 2y$ $\qquad$ $y^2 - 2x - 5 = 0$

plug in

$y^2 - 4y - 5 = 0$

$y_{1,2} = \frac{4 \pm \sqrt{16+20}}{2}$ $<$ $5 \rightsquigarrow x = 2\cdot 5 = 10$; $-1 \rightsquigarrow x = 2\cdot(-1) = -2$

$\rightsquigarrow$ critical points are $(10,5)$ and $(-2,-1)$

2. Find f_{xx}, D:

$f_{xx} = \frac{\partial}{\partial x}(2x - 4y) = 2$ $\qquad$ $f_{xy} = \frac{\partial}{\partial y}(2x - 4y) = -4$

$f_{yy} = \frac{\partial}{\partial y}(2y^2 - 4x - 10) = 4y$ $\qquad$ $D = 2\cdot 4y - (-4)^2 = 8y - 16$

3. Test the critical pts:

for $(10,5)$:

$f_{xx} = 2 > 0$

$D = 40 - 16 = 24 > 0$

local min

for $(-2,-1)$:

$f_{xx} = 2 > 0$

$D = -8 - 16 = -24 < 0$

saddle point

Exercise 3. Find all the local maxima, minima and saddle points of the function

$$f(x,y) = \frac{3}{2}x^4 - yx^2 + 20x^2 + \frac{1}{2}y^2 - 3.$$

1. critical pts:

$\frac{\partial f}{\partial x} = 6x^3 - 2xy + 40x \qquad \frac{\partial f}{\partial y} = -x^2 + y$

$6x^3 - 2xy + 40x = 0$ & $-x^2 + y = 0$

plug in

$y = x^2$

$6x^3 - 2x^3 + 40x = 0$

$4x^3 + 40x = 0$

$x^3 + 10x = 0$

$x(x^2+10) = 0$ → the only critical point is $(0,0)$.

$x = 0$ or $x^2 + 10 = 0$

$y = 0^2 = 0$ (no solution)

2. Find f_{xx}, D

$f_{xx} = \frac{\partial}{\partial x}(6x^3 - 2xy + 40x) = 18x^2 - 2y + 40 \qquad f_{xy} = \frac{\partial}{\partial y}(6x^3 - 2xy + 40x) = -2x$

$f_{yy} = \frac{\partial}{\partial y}(-x^2 + y) = 1 \qquad D = (18x^2 - 2y + 40)\cdot 1 - (-2x)^2 =$

$= 18x^2 - 2y + 40 - 4x^2 = 14x^2 - 2y + 40$

3. Test critical points

Test $(0,0)$: $f_{xx} = 40 > 0$,

$D = 14\cdot 0 - 2\cdot 0 + 40 = 40 > 0$ ⇒ local minima