MA 16020 Lesson 22: Chain rule for multivariate functions

When $z = f(x, y)$ is a function of x and y, and $x = g(t), y = h(t)$ are further functions of a common variable t, the overall function $z = f(g(t), h(t))$ is a function of one variable. While its derivative $\mathrm{d}z/\mathrm{d}t$ can be computed directly, it is often benefitial to use the *chain rule for functions of two variables:*

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

$$\left(\text{or: } \frac{dz}{dt}(t) = \frac{\partial f}{\partial x}(g(t), h(t))\cdot g'(t) + \frac{\partial f}{\partial y}(g(t), h(t))\cdot h'(t) \right)$$

Example: If $z = x/y$, $x = \ln(t)$, $y = t^3 + t^2 + 1$, we may compute $\mathrm{d}z/\mathrm{d}t$

(a) using the chain rule:

$$\frac{dz}{dt} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right)\cdot(\ln(t))' + \frac{\partial}{\partial y}\left(\frac{x}{y}\right)\cdot(t^3+t^2+1)' = \frac{1}{y}\cdot\frac{1}{t} - \frac{x}{y^2}(3t^2+2t)$$

(b) directly: $z = \frac{x}{y} = \frac{\ln(t)}{t^3+t^2+1}$,

$$\frac{dz}{dt} = \frac{\frac{1}{t}(t^3+t^2+1) - \ln(t)\cdot(3t^2+2t)}{(t^3+t^2+1)^2}$$

Let us verify that the result is the same:

$$\frac{1}{y}\cdot\frac{1}{t} - \frac{x}{y^2}(3t^2+2t) = \frac{1}{t^3+t^2+1}\cdot\frac{1}{t} - \frac{\ln(t)}{(t^3+t^2+1)^2}\cdot(3t^2+2t)$$

$$= \frac{\frac{1}{t}\cdot(t^3+t^2+1) - \ln(t)(3t^2+2t)}{(t^3+t^2+1)^2} \checkmark$$

Exercise 1. Compute $\frac{dz}{dt}$ when $z = 3x^2y^3$, $x = \sin(3t+1)$ and $y = e^{2t} - 2$

$$\frac{dz}{dt} = \frac{\partial}{\partial x}(3x^2 \cdot y^3) \cdot (\sin(3t+1))' + \frac{\partial}{\partial y}(3x^2y^3) \cdot (e^{2t}-2)' =$$

$$= 6x \cdot y^3 \cdot (3 \cdot \cos(3t+1)) + 9x^2y^2 \cdot (2 \cdot e^{2t}) =$$

$$= \underline{\underline{18xy^3 \cdot \cos(3t+1) + 18x^2y^2 \cdot e^{2t}}}$$

Exercise 2. Evaluate $\frac{dz}{dt}$ at $t = 2$ when $z = \sin(xy)$, $x = \frac{\pi t^2}{4}$ and $y = \frac{t}{4}$.

$$\frac{dz}{dt} = \frac{\partial}{\partial x}(\sin(xy)) \cdot \left(\frac{\pi \cdot t^2}{4}\right)' + \frac{\partial}{\partial y}(\sin(xy)) \cdot \left(\frac{t}{4}\right)' =$$

$$= (y\cos(xy)) \cdot \left(\frac{\pi t}{2}\right) + (x\cos(xy)) \cdot \frac{1}{4} =$$

$$= \frac{\pi}{2} y \cdot \cos(xy) \cdot t + \frac{1}{4} x \cos(xy) = \cos(xy)\left(\frac{\pi}{2} yt + \frac{1}{4}x\right)$$

when $\underline{t=2}$, $x = \frac{\pi t^2}{4} = \frac{\pi \cdot 2^2}{4} = \underline{\pi}$,

$y = \frac{t}{4} = \frac{2}{4} = \underline{\frac{1}{2}}$

$$\rightarrow \frac{dz}{dt}(2) = \cos\left(\pi \cdot \frac{1}{2}\right) \cdot \left(\frac{\pi}{2} \cdot \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot \pi\right)$$

$$= 0 \cdot \left(\frac{3}{4}\pi\right)^2 = \underline{\underline{0}}$$

Exercise 3. The number of units of a certain product sold per month is given by the function

$$S(x, y) = 3y^2 + 2yx + x,$$

where x is the amount spent on advertising per month (in thousands of dollars) and y is the amount spent on distribution per month (in thousands of dollars). Given that t months from now, the monthly amount spent on advertising is $15 + t$ thousands of dollars and the monthly amount spent on distribution is $t^2 + t + 3$ thousands of dollars, find the rate of change of the number of units sold per month 4 months from now.

$x(t) = 15 + t,$

$y(t) = t^2 + t + 3$

Want: $\frac{dS}{dt}(4)$

$$\frac{dS}{dt} = \frac{\partial}{\partial x}\left(3y^2 + 2yx + x\right) \cdot (15+t)' + \frac{\partial}{\partial y}\left(3y^2 + 2yx + x\right) \cdot (t^2 + t + 3)'$$

$$= (2y + 1) \cdot 1 + (6y + 2x) \cdot (2t + 1)$$

$\cancel{= 2y + 4t + 12yt}$

when $t = 4$, $x(4) = 15 + 4 = 19,$

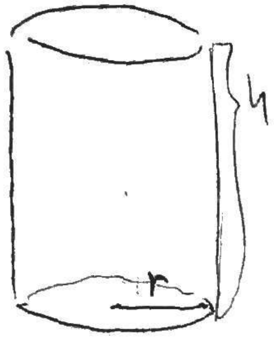
$y(4) = 4^2 + 4 + 3 = 16 + 7 = 23$

$$\rightarrow \frac{dS}{dt}(4) = (2 \cdot 23 + 1) + (6 \cdot 23 + 2 \cdot 19) \cdot (2 \cdot 4 + 1) =$$

$$= 47 + (138 + 38) \cdot 9 = 47 + 1584$$

$$= 1631$$ ~~thousand of dollars~~ units/monthly month^2

Exercise 4. The radius of the base of a cylinder is decreasing at the rate 3 mm/s while its height is increasing at the rate 7 mm/s. What is the rate of change of the volume of the cylinder at the moment when the radius is 40 mm and the height is 85 mm?



V = Volume of the cylinder $= V(r,h) = \pi r^2 \cdot h$

Know: $\frac{dr}{dt} = -3$, $\frac{dh}{dt} = 7$

Want: $\frac{dV}{dt}$ when $r = 40$, $h = 85$

$$\frac{dV}{dt} = \frac{\partial}{\partial r}\left(\pi r^2 \cdot h\right) \cdot \frac{dr}{dt} + \frac{\partial}{\partial h}\left(\pi r^2 h\right) \cdot \frac{dh}{dt} =$$

$$= 2\pi r h \cdot (-3) + \pi r^2 \cdot 7$$

$r = 40$, $h = 85$:

$$\frac{dV}{dt} = 2 \cdot \pi \cdot 40 \cdot 85 \cdot (-3) + \pi \cdot 40^2 \cdot 7$$

$$= \pi(-20\,400 + 11\,200) = \underline{\underline{-9200\pi \ mm^3/s}}$$

$$\approx -28\,902.65 \ mm^3/s$$

$\rightarrow$ The volume of the cylinder ~~is decrea~~ at that moment is decreasing by approx $28\,902.65 \ mm^3/s$.