MA 16020 Lesson 20: Higher partial derivatives

When $z = f(x, y)$ is a function of two variables, so are the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Upon taking partial derivatives of these two functions, we obtain four *second-order partial derivatives* of f:

1) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{(\partial x)^2} = f_{xx}$,   3) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$,

2) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$,   4) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{(\partial y)^2} = f_{yy}$.

Fact: While this is not true in full generality, for all functions we encounter, we have

$$f_{xy} = f_{yx}.$$

Exercise 1. Compute all the second-order partial derivatives for

$$f(x, y) = y^3 x e^{xy}.$$

$f_x = y^3 \cdot e^{xy} + y^3 x \cdot (y e^{xy}) = y^3 e^{xy} + y^4 x e^{xy} = (y^3 + y^4 x) e^{xy}$

$f_y = 3y^2 x e^{xy} + y^3 x (x e^{xy}) = 3y^2 x e^{xy} + y^3 x^2 e^{xy} = (3y^2 x + y^3 x^2) e^{xy}$

$f_{xx} = \frac{\partial}{\partial x}(f_x) = y^4 e^{xy} + (y^3 + y^4 x)(y e^{xy}) = y^4 e^{xy} + (y^4 + y^5 x) e^{xy} = (2y^4 + y^5 x) e^{xy}$

$f_{xy} = \frac{\partial}{\partial y}(f_x) = (3y^2 + 4y^3 x) e^{xy} + (y^3 + y^4 x)(x e^{xy}) = (3y^2 + 4y^3 x + y^3 x + y^4 x^2) e^{xy}$
$= (3y^2 + 5y^3 x + y^4 x^2) e^{xy}$

$f_{yy} = \frac{\partial}{\partial y}(f_y) = (6yx + 3y^2 x^2) e^{xy} + (3y^2 x + y^3 x^2)(x e^{xy}) =$
$= (6yx + 6y^2 x^2 + y^3 x^3) e^{xy}$

$f_{yx} = \frac{\partial}{\partial x}(f_y) = (3y^2 + 2y^3 x) e^{xy} + (3y^2 x + y^3 x^2)(y e^{xy}) = (3y^2 + 5y^3 x + y^4 x^2) e^{xy} = f_{xy}$

Exercise 2. Compute f_{uu} and f_{uv} for

$$f(u,v) = \sqrt{u^2 + v^4 + 2}\,.$$

$$f_u = \frac{\cancel{2u}\,1}{2\sqrt{u^2+v^4+2}} \cdot \cancel{2u} = \frac{u}{\sqrt{u^2+v^4+2}},$$

$$f_{uu} = \frac{1\cdot\sqrt{u^2+v^4+2} - \frac{u}{\sqrt{u^2+v^4+2}}\cdot u}{\left(\sqrt{u^2+v^4+2}\right)^2} = \frac{1}{\sqrt{u^2+v^4+2}} - \frac{u^2}{\left(\sqrt{u^2+v^4+2}\right)^3}$$

$$f_{uv} = \frac{\partial}{\partial v}\left(u \cdot (u^2+v^4+2)^{-\frac{1}{2}}\right) = u\cdot\left(-\frac{1}{2}\right)\cdot(u^2+v^4+2)^{-\frac{3}{2}}\cdot 4v^3 =$$

$$= \frac{-2uv^3}{\left(\sqrt{u^2+v^4+2}\right)^3}$$

Exercise 3. Compute $f_{yy}(1,2)$ when

$$f(x,y) = \ln(2x^3 + 3xy + y)\,.$$

$$f_y = \frac{1}{2x^3+3xy+y}\cdot(3x+1),$$

$$f_{yy} = (3x+1)\cdot(-1)\cdot\frac{1}{(2x^3+3xy+y)^2}\cdot(3x+1) = -\frac{(3x+1)^2}{(2x^3+3xy+y)^2}$$

$$f_{yy}(1,2) = \frac{-(3+1)^2}{(2\cdot1^3+3\cdot1\cdot2+2)^2} = -\frac{4^2}{10^2} = -\frac{16}{100} = -0{,}16$$

Exercise 4. Compute $f_{xy}(1,3)$ when

$$f(x,y) = 3y^2 \ln(x) + \underbrace{\frac{\sqrt{e^{3x} + \ln(x^3+2)}}{5\sqrt[3]{\sin^2(x-4)+1}}}_{(*)} + 2yx^3 \,.$$

f_x ... hard because of (*)

⇝ use $f_{xy} = f_{yx}$:

$f_y = 6y \cdot \ln(x) + 0 + 2x^3$,

$f_{yx} = \frac{6y}{x} + 0 + 6x^2 = f_{xy}$.

$\rightarrow \quad f_{xy}(1,3) = \frac{18}{1} + 6 = 24$

Exercise 5. Compute all the second–order partial derivatives of

$$f(u,v) = \cos(3u)\sin(4uv) \,.$$

$f_u = (-3\sin 3u) \cdot \sin(4uv) + \cos(3u) \cdot (4v \cos(4uv)) = -3\sin(3u)\sin(4uv) + 4v\cos(3u)\cos(4uv)$

$f_v = \cos(3u) \cdot (4u \cos(4uv))$

$f_{uu} = -9\cos(3u) \cdot \sin(4uv) - 3\sin(3u)(4v\cos(4uv)) + 4v(-3\sin(3u)\cos(4uv) - 4v\cos(3u)\sin(4uv))$

$= -9\cos(3u) \cdot \sin(4uv) - 12v\sin(3u)\cos(4uv) - 12v\sin(3u)\cos(4uv) - 16v^2\cos(3u)\sin(4uv)$

$= -9\cos(3u)\sin(4uv) - 24v\sin(3u)\cos(4uv) - 16v^2\cos(3u)\sin(4uv)$

$= (-9 - 16v^2)\cos(3u)\sin(4uv) - 24v\sin(3u)\cos(4uv)$

$f_{vv} = (4u \cdot \cos(3u))(-4u\sin(4uv)) = -16u^2\cos(3u)\sin(4uv)$

$f_{vu} = f_{uv} = \frac{\partial}{\partial v}(-3\sin(3u)\sin(4uv) + 4v\cos(3u)\cos(4uv)) =$

$= -3\sin(3u) \cdot (4u\cos(4uv)) + 4\cos(3u)\cos(4uv) + 4v\cos(3u)(-4u\sin(4uv))$

$= -12u\sin(3u)\cos(4uv) + 4\cos(3u)\cos(4uv) - 16uv\cos(3u)\sin(4uv)$