# MATH 16020 Lesson 1A: Integration by Substitution I

Spring 2021

**Warm-up.** Find the derivative of $2(3x+4)^{10}$.

By Chain Rule, $y' = 2(10(3x+4)^9)\cdot 3 = \boxed{60(3x+4)^9}$

$\Rightarrow$ Antiderivative of $60(3x+4)^9$ is $2(3x+4)^{10}+C$, or $\boxed{\int 60(3x+4)^9\,dx = 2(3x+4)^{10}+C}$

BUT how do we find antiderivative if we start with

$\int 60(3x+4)^9\,dx$ ? Apply substitution!

**Idea behind substitution:** Undo chain rule, so consider expression inside power of 9 $(3x+4)$ and let $u = 3x+4 \Rightarrow \frac{du}{dx} = 3$, or $\boxed{du = 3dx}$ ← Will be using this form with substitution

Now, solve for $dx$ to get $\frac{du}{3} = dx$ to get

$$\int 60(3x+4)^9\,dx = \int 60(u)^9\,\frac{du}{3} = \int 20u^9\,du = 20\,\frac{u^{10}}{10} + C = 2u^{10} + C = \boxed{2(3x+4)^{10}+C}$$

MUST be in terms of u and du NO x or dx!

Original integral in terms of x, so answer must be in terms of x also!

<u>Note</u>: For substitution problems where a linear function ($mx+b$ or $mt+b$, depends on problem) is "inside" another function (e.g., $\cos(3x+2)$, $e^{-x}$), let u be the linear function.

**Example 1.** Evaluate: $\int x^2\sqrt{31-5x^3}dx$

How to choose u here? Choose part of expression inside integral whose derivative equals a constant multiple of another part (in more general setting).

Using this rule, note derivative of $31-5x^3$ is $-15x^2$, a constant multiple of $x^2$

$\Rightarrow$ Let $u=31-5x^3$

$\Rightarrow du=-15x^2dx \Rightarrow dx=\frac{du}{-15x^2}$

$\Rightarrow \int x^2\sqrt{31-5x^3}dx = \int \cancel{x^2}\sqrt{u}\frac{du}{-15\cancel{x^2}} = \int -\frac{1}{15}u^{1/2}du = \frac{-1}{15}\cdot\frac{2}{3}u^{3/2}+C$

$= \boxed{\frac{-2}{45}(31-5x^3)^{3/2}+C}$

**Example 2.** Evaluate: $\int e^{x+e^x}dx$

$\int e^{x+e^x}dx = \int e^x e^{e^x}dx = \int \cancel{e^x}e^u\frac{du}{\cancel{e^x}} = \int e^u du = e^u + C = \boxed{e^{e^x}+C}$

$u=e^x$
$du=e^x dx$
$\Rightarrow dx=\frac{du}{e^x}$

**Example 3.** Find the function $f(x)$ whose tangent line has the slope $\tan(x)$ for $x$ in the domain of $\tan(x)$ whose graph passes through the point $(2\pi, 6)$.

Recall: Derivative $f'(x)$ gives slope of $f(x)$ at $x$.

$\Rightarrow$ Given $\begin{cases} f'(x) = \tan(x) \\ f(2\pi) = 6 \end{cases}$, Find $f(x)$.

$u = \cos(x)$
$du = -\sin(x)dx$
$\Rightarrow dx = \dfrac{du}{-\sin(x)}$

$\Rightarrow$ First, find general sol'n.

$$f(x) = \int f'(x)dx = \int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = \int \frac{\cancel{\sin(x)}}{u} \cdot \frac{du}{-\cancel{\sin(x)}} = \int \frac{-1}{u}du$$

$$= -\ln(|u|) + C$$

$$= \boxed{-\ln|\cos(x)| + C}$$

$\cancel{u = \sin(x)}$
$\cancel{du = \cos(x)dx}$
$\cancel{\Rightarrow dx = \dfrac{du}{\cos(x)}}$

$\cancel{\displaystyle\int \frac{u}{\cos(x)} \frac{du}{\cos(x)}}$

$u = \sin(x)$ no good!

$\Rightarrow f(x) = -\ln|\cos(x)| + C$

$\Rightarrow f(2\pi) = -\ln|\cos(2\pi)| + C = -\underbrace{\ln(1)}_{0} + C = \boxed{C = 6}$

$\Rightarrow \boxed{f(x) = -\ln|\cos(x)| + 6}$

**Example 4.** Suppose a microwave heats a brownie in such a way that the temperature of the brownie increases at a rate of:

initial $\longleftrightarrow t=0$

$$T'(t) = 54t^2e^{-3t^3}\ ^\circ\mathrm{F/min}$$

If the brownie has temperature 30°$F$ going into the microwave, how long should the microwave heat the brownie so the brownie has temperature 33°$F$? Round answer to nearest hundreth.

Let $T(t)$ = temp. function @ time $t$ (°F)

$\begin{cases} T(0) = 30 \\ T'(t) = \ \ \end{cases} \Rightarrow$ Find $T(t)$, then solve $T(t)=33$.

$$T(t) = \int T'(t)\,dt = \int 54t^2 e^{-3t^3}\,dt = \int 54\cancel{t^2} e^{u}\left(\frac{du}{-9\cancel{t^2}}\right) = \int -6e^u\,du$$

$$u = -3t^3$$
$$du = -9t^2\,dt$$
$$\Rightarrow dt = \frac{du}{-9t^2}$$

$$= -6e^u + C$$

$$= \boxed{-6e^{-3t^3} + C} = T(t)$$

$$T(0) = -6e^0 + C = -6 + C = 30$$
$$\Rightarrow C = 36$$
$$\Rightarrow \boxed{T(t) = -6e^{-3t^3} + 36}$$

$$33 = -6e^{-3t^3} + 36$$
$$\Rightarrow \frac{1}{2} = e^{-3t^3}$$
$\ln(-)$
$$\Rightarrow \ln(1/2) = -3t^3$$
$$\Rightarrow t = \sqrt[3]{\frac{\ln(1/2)}{-3}} \approx 0.6136227 \text{ min.}$$
$$\approx 36.8 \quad \text{sec.}$$
$\times 60$ (in case final answer must be in sec.)