MA 16020 Lesson 19: Partial derivatives

For a function $z = f(x, y)$ of two variables, we have two ways to take derivatives:

The (first) **partial derivative** $\frac{\partial f}{\partial x}$ (or $\frac{\partial z}{\partial x}$, f_x) describes the rate of change of z as $\underline{x}$ changes and $\underline{y}$ remains constant. It is computed as a derivative of f as a function of $\underline{x}$ where we treat the variable $\underline{y}$ as a constant.

The (first) **partial derivative** $\frac{\partial f}{\partial y}$ (or $\frac{\partial z}{\partial y}$, f_y) describes the rate of change of z as $\underline{y}$ changes and $\underline{x}$ remains constant. It is computed as a derivative of f as a function of $\underline{y}$ where we treat the variable $\underline{x}$ as a constant.

Example: Compute the first partial derivatives of the function

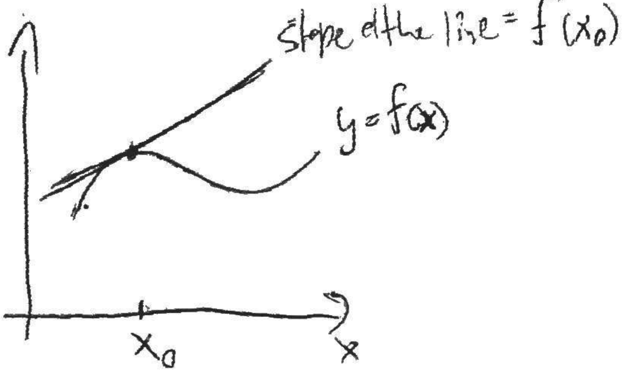
$$f(x, y) = x^2 + xy + 5\ln(y) .$$

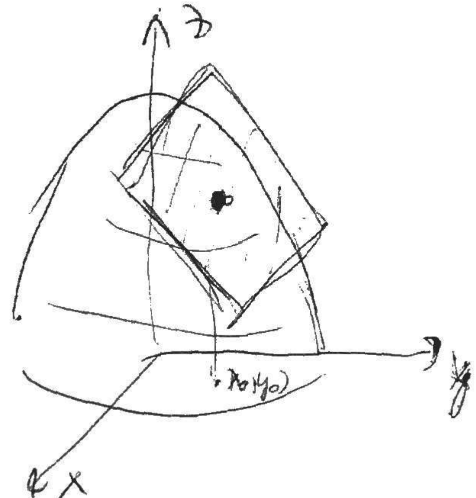
$$\frac{\partial f}{\partial x} = 2x + y + 0 = \underline{\underline{2x + y}}$$

$$\frac{\partial f}{\partial y} = 0 + x + \frac{5}{y} = \underline{\underline{x + \frac{5}{y}}}$$

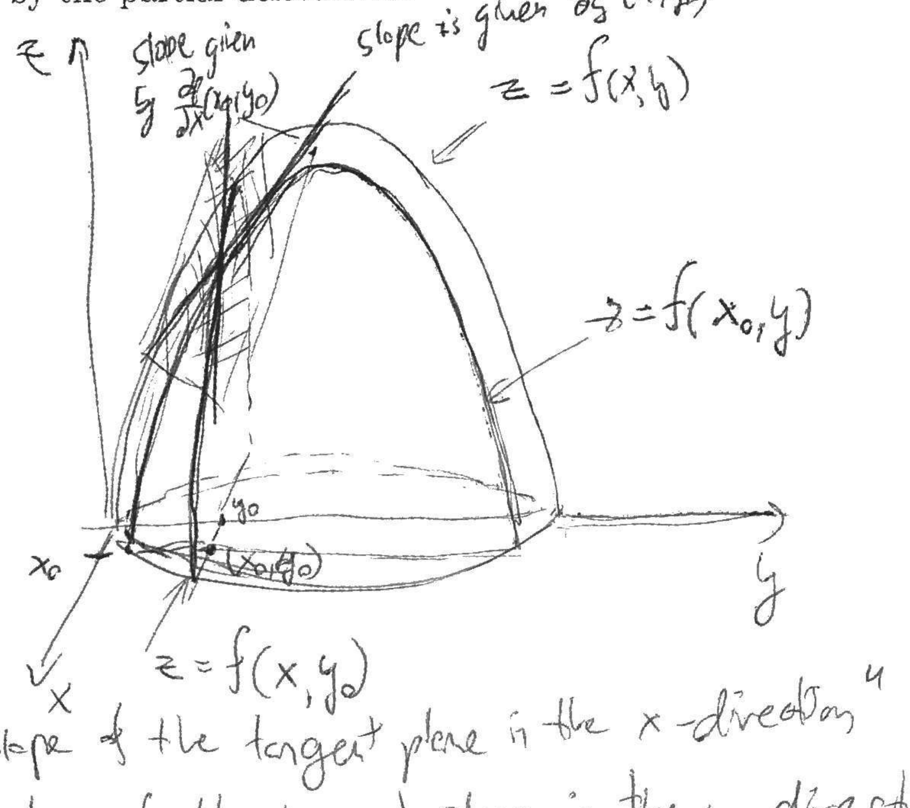
Recall: The graph of a function of one variable $f(x)$ at a given x_0 has a tangent line, whose slope is dictated by the derivative of f at x_0, $f'(x_0)$

The graph of a function of two variables $f(x, y)$ at a given (x_0, y_0) has a tangent ______________.





It can be determined by the partial derivatives:



$\frac{\partial f}{\partial x}(x_0, y_0) =$ "the slope of the tangent plane in the x-direction"

$\frac{\partial f}{\partial y}(x_0, y_0) =$ "the slope of the tangent plane in the y-direction"

Exercise 1. Compute $f_x \cdot f_y$ when

$$f(x,y) = \frac{3xy}{\sqrt{xy-1}}\,.$$

$$\frac{\partial f}{\partial x} = \frac{3y\cdot\sqrt{xy-1} - \frac{1}{2\sqrt{xy-1}}\cdot y\cdot 3xy}{\left(\sqrt{xy-1}\right)^2} = \frac{3y\sqrt{xy-1} - \frac{3xy^2}{2\sqrt{xy-1}}}{xy-1} =$$

$$= \frac{3y}{\sqrt{xy-1}} - \frac{3xy^2}{2\left(\sqrt{xy-1}\right)^{3/2}}$$

$$\frac{\partial f}{\partial y} = \frac{3x\sqrt{xy-1} - \frac{1}{2\sqrt{xy-1}}\cdot x\cdot 3xy}{\left(\sqrt{xy-1}\right)^2} = \frac{3x\sqrt{xy-1} - \frac{3x^2y}{2\sqrt{xy-1}}}{xy-1} = \frac{3x}{\sqrt{xy-1}} - \frac{3x^2y}{2\left(\sqrt{xy-1}\right)^2}$$

Exercise 2. Compute $f_x(1,3)$ when

$$f(x,y) = \frac{\ln(3xy+3)}{x+y}\,.$$

$$f_x(x,y) = \frac{\frac{1}{3xy+3}\cdot(3y)\cdot(x+y) - \ln(3xy+3)\cdot 1}{(x+y)^2} =$$

$$= \frac{3y}{(3xy+3)(x+y)} - \frac{\ln(3xy+3)}{(x+y)^2} = \frac{y}{(xy+1)(x+y)} - \frac{\ln(3xy+3)}{(x+y)^2}$$

$$f_x(1,3) = \frac{3}{(3+1)(3+1)} - \frac{\ln(3\cdot1\cdot3+3)}{(3+1)^2} = \frac{3}{16} - \frac{\ln(12)}{16}$$

Exercise 3. The pressure (in Pa) of certain gas in a container is decribed by the equation

$$P = 50\frac{T}{V}$$

where T is the temperature of the gas (in °K) and V is the volume of the container (in m^3). If the temperature of the gas is 320°K and the gas is kept in a container of volume $5\,m^3$, find the rate of change of the pressure both with respect to the change of temperature and with respect to the change of volume.

The rate of change of pressure with respect to change of temperature (assuming volume to be constant),

$$= \frac{\partial P}{\partial T} = 50 \cdot \frac{1}{V}$$

The rate of change in the given conditions ($V = 5\,m^3, T = 320°K$)

$$= \frac{\partial P}{\partial T}(320, 5) = 50 \cdot \frac{1}{5} = 10 \text{ Pa}/°K$$

The rate of change of pressure wrt volume (assuming constant temperature) $= \frac{\partial P}{\partial V}$

$$\frac{\partial P}{\partial V} = 50 \cdot T\left(-\frac{1}{V^2}\right) = -50 \frac{T}{V^2}$$

The rate at the given conditions is

$$\frac{\partial P}{\partial V}(320, 5) = -50 \cdot \frac{320}{25} = -640 \text{ Pa}/m^3$$

Exercise 4. A company makes products A and B. If it produces x units of product A and y units of product B, the expected revenue is

$$R(x, y) = 5x + 10y + 3xy \, .$$

If the company revenues makes 15 units of product A and 10 units of product B, find the marginal ~~profits~~ (=rates of change with respect to change of production of product A and product B, resp.)

The task is:

Compute $\frac{\partial R}{\partial x}(15,10)$

marginal revenues

$\frac{\partial R}{\partial y}(15,10)$

$$\frac{\partial R}{\partial x} = 5 + 0 + 3y$$

$$\frac{\partial R}{\partial x}(15,10) = 5 + 3 \cdot 10 = \underline{\underline{35}}$$

$$\frac{\partial R}{\partial y} = 0 + 10 + 3x$$

$$\frac{\partial R}{\partial y}(15,10) = 10 + 3 \cdot 15 = \underline{\underline{55}}$$