MA 16020 Lesson 18: Functions of several variables

A function of *one variable* takes as an input a number

and produces as an output a number.

A graph of such a function is a curve in the xy-plane.

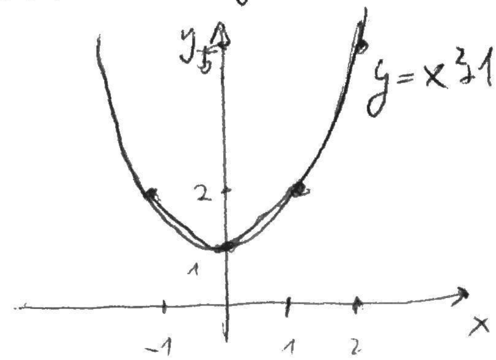
Example: $f(x) = x^2 + 1$

$f(0)=1$ (input: 0 output: 1)

$f(1)=2$ (in: 1 out: 2)

$f(-1)=2$

$f(2)=5$, etc.



A function of *two variables* takes as an input a pair of ~~number~~ numbers / point in the xy-plane

and produces as an output a number.

A graph of such a function is a surface in xyz-space.

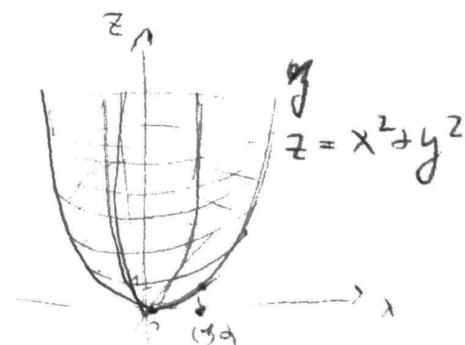
Example: $f(x, y) = x^2 + y^2$

$f(0,0)=0$ (input: (0,0) output: 0)

$f(1,0)=1$ (in: (1,0) out: (0,1))

$f(0,1)=1$ (in: (0,1) out: 1

$f(2,0)=4$ ... etc



A function of *three variables* takes as an input a triple of numbers / a point in xyz-space

and produces as an output a number.

Example: $f(x, y, z) = 2x^2 - xyz^2$

$f(0,0,0)=0$

$f(1,0,0)=2\cdot 1^2-1\cdot 0\cdot 0^2=2, \dots$

... and so on.

(We will stick to functions of two variables for the most part.)

$f(x,y)$

Just as for functions of one variable, we consider for functions of two (or three etc.) variables their

domain = all points (x,y) in the xy-plane for which $f(x,y)$ is defined

... can be a complicated region in the plane, usually we use the set notation $\{(x,y) \mid \ldots\}$

and range = all the values that the function $f(x,y)$ can attain, ~~i.e.~~

Example: Find the domain and the range of the function

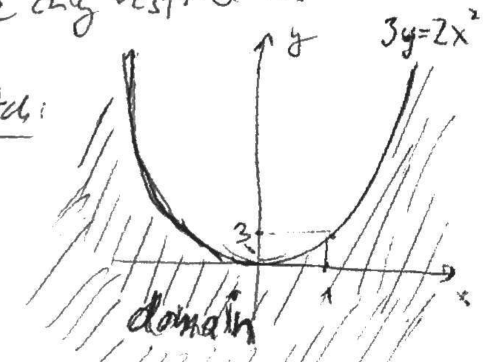
$$f(x,y) = \sqrt{2x^2 - 3y} + 5\,.$$

Domain: for $\sqrt{\ldots}$ to be defined, we need the argument to be non-negative

$\leadsto$ $2x^2 - 3y \geq 0$. This is the only restriction.

So Domain $= \{(x,y) \mid 2x^2 - 3y \geq 0\}$ Sketch:

$= \{(x,y) \mid 2x^2 \geq 3y\}$



Range: $\sqrt{}$ is always ≥ 0

$\Rightarrow$ $\sqrt{2x^2 - 3y} + 5 \geq 5$ no matter what x, y are

So range is contained in $[5, \infty)$

On the other hand, every value from $[5,\infty)$ is attainable

(given $z \geq 5$, then $z = f(x,y)$ e.g. for $x = \frac{z-5}{\sqrt{2}}$ and $y = 0$:

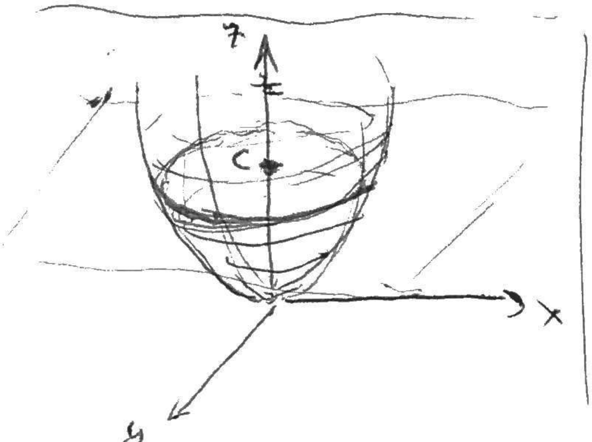
$f\left(\frac{z-5}{\sqrt{2}}, 0\right) = \sqrt{2 \cdot \left(\frac{z-5}{\sqrt{2}}\right)^2 - 0} + 5 = \sqrt{(z-5)^2} + 5 = z - 5 + 5 = z$)

$\Rightarrow$ $\boxed{\text{Range} = [5, \infty)}$

The graph of a function $f(x, y)$ of two variables is sometimes analyzed via the so-called

level curves = Curves in the xy-plane defined by $f(x,y) = C$,
C some constant

= curves consisting of all points (x,y) with equal values of $f(x,y)$



Example: Find and sketch the level curves of the function

$$f(x, y) = e^{x^2+y^2}$$

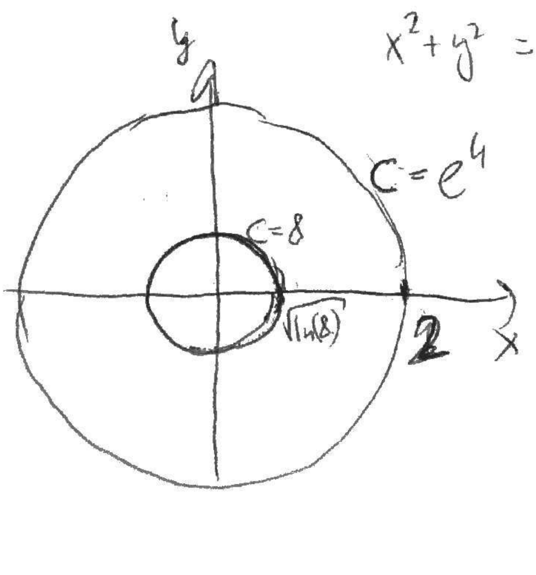
for $C = 8$ and $C = e^4$.

$C = e^4$: $\quad e^{x^2+y^2} = e^4 \quad / \ln(-)$

$x^2+y^2 = 4$ circle of radius 2 centered at $(0,0)$

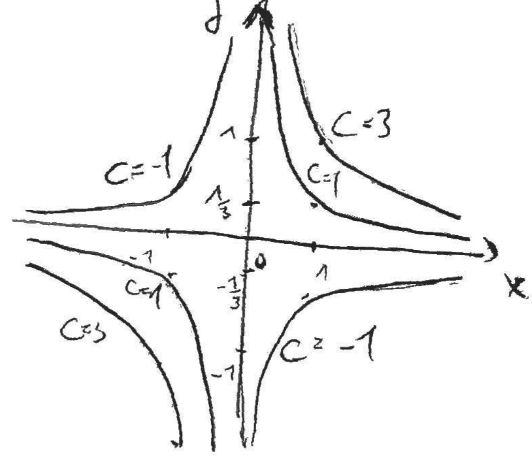
$C = 8$ $\quad e^{x^2+y^2} = 8 \quad / \ln(-)$

$x^2+y^2 = \ln(8)$ circle of radius $\sqrt{\ln(8)}$ centered at $(0,0)$



Exercise 1. Describe the level curves of

$$f(x,y) = 3x^3y.$$



We pick some c's on our own:

$\underline{c = 1}$ $\quad 1 = 3x^3y \rightsquigarrow y = \frac{1}{3x^3}$

$\underline{c = -1}$ $\quad -1 = 3x^3y \rightsquigarrow y = \frac{-1}{3x^3}$

$\underline{c = 3}$; $\quad 3 = 3x^3y \rightsquigarrow y = \frac{3}{3x^3} \left(= \frac{1}{x^3}\right)$

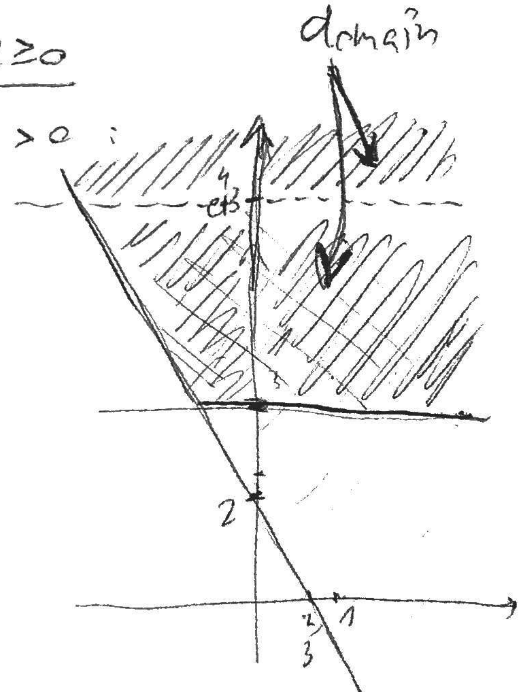
(!) $\underline{c = 0}$; $\quad 0 = 3x^3y \rightsquigarrow \underline{x = 0 \text{ or } y = 0}$

coordinate axes

$\Rightarrow$ level curves are (generally) ~~rational fun~~ graphs of rational functions symmetric about the origin

Exercise 2. Find the domain of

$$f(x,y) = \frac{\sqrt{3x+y-1}\cdot\sqrt{3x+y-2}}{\ln(y-3)-4}.$$



Restrictions: • arguments of "$\sqrt{\ldots}$" need to be ≥ 0:

$3x+y-1 \geq 0$, $\quad \underline{3x+y-2 \geq 0}$

• argument of $\ln(\cdot)$ needs to be > 0:

$y - 3 > 0 \rightsquigarrow \underline{y > 3}$

• denominator needs to be $\neq 0$

$\ln(y-3) - 4 \neq 0$

$\ln(y-3) \neq 4$

$y - 3 \neq e^4$

$\underline{y \neq e^4 + 3}$

redundant

$\rightarrow$ Domain $= \left\{ (x,y) \,\middle|\, 3x+y-1 \geq 0,\ 3x+y-2 \geq 0,\ y > 3,\ y \neq e^4+3 \right\}$

$= \left\{ (x,y) \,\middle|\, 3x+y-2 \geq 0,\ y > 3,\ y \neq e^4+3 \right\}$