MA 16020 Lesson 17: Geometric series II

Recall (geometric series):

A geometric series is a series of the form: $\sum_{n=0}^{\infty} kq^n = k + kq + kq^2 + \dots$

The series conveges if and only if: $|q| < 1$ ⇔ $-1 < q < 1$

In that case, the sum of the series is: $\boxed{S = \frac{k}{1-q}}$

Exercise 1. Write

$$\frac{49}{25} - \frac{7}{5} + 1 - \frac{5}{7} + \frac{25}{49} + \dots \quad = (*)$$

in a compact form, and find its sum.

$\frac{49}{25}$

$\frac{49}{25} \cdot \left(-\frac{5}{7}\right) = -\frac{7}{5}$

$\frac{49}{25} \cdot \left(-\frac{5}{7}\right)^2 = \left(-\frac{7}{5}\right) \cdot \left(-\frac{5}{7}\right) = 1$

$\frac{49}{25} \cdot \left(-\frac{5}{7}\right)^3 = 1 \cdot \left(-\frac{5}{7}\right) = -\frac{5}{7},$

$-\frac{49}{25} \cdot \left(-\frac{5}{7}\right)^4 = \left(-\frac{5}{7}\right)\left(-\frac{5}{7}\right) = \frac{25}{49}$

$\rightarrow (*) = \sum_{n=0}^{\infty} \left(\frac{49}{25}\right) \cdot \left(-\frac{5}{7}\right)^n =$

$= \frac{49/25}{1 - \left(-\frac{5}{7}\right)} = \frac{\frac{49}{25}}{1 + \frac{5}{7}} = 68.6$

Exercise 2. Compute

$$\sum_{n=2}^{\infty} \frac{2}{3^{2n}} = \sum_{n=2}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^{2n} = \sum_{n=2}^{\infty} 2\left(\left(\frac{1}{3}\right)^2\right)^n =$$

$$= \sum_{n=2}^{\infty} 2 \cdot \left(\frac{1}{9}\right)^n = 2 \cdot \left(\frac{1}{9}\right)^2 + 2 \cdot \left(\frac{1}{9}\right)^3 + 2\left(\frac{1}{9}\right)^4 + \dots$$

$$= \underbrace{2 \cdot \left(\frac{1}{9}\right)^2}_{k} + \underbrace{2 \cdot \left(\frac{1}{9}\right)^2}_{k} \underbrace{\left(\frac{1}{9}\right)^1}_{q^1} + \underbrace{2\left(\frac{1}{9}\right)^2}_{k} \cdot \underbrace{\left(\frac{1}{9}\right)^2}_{q^2} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{2}{81} \left(\frac{1}{9}\right)^n 1 = \frac{(2/81)}{1 - \frac{1}{9}} = \frac{(2/81)}{\frac{8}{9}} = \frac{1}{4 \cdot 9} = \frac{1}{36}$$

Exercise 3. A forest restoration organization plants 100 new trees each year. At the same time, it is expected that each year, 8% of all growing trees die due to various causes. Assuming that this effort goes on indefinitely, what is the expected eventual number of trees in the forest right after a round of re-planting? [Round to the nearest integer.]

8% of trees die $\Rightarrow$ 92% of trees survive

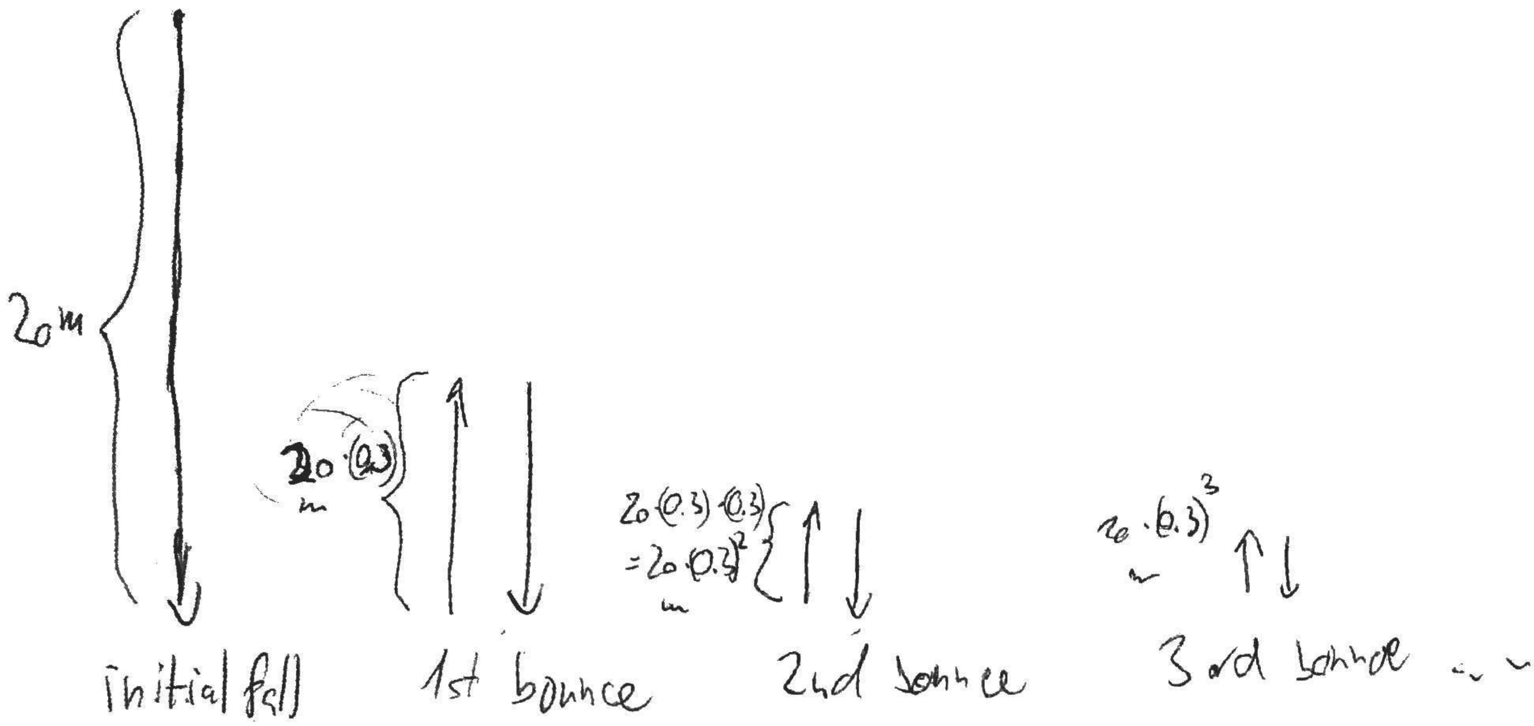
Eventual number of trees =

$$= 100 + \underbrace{100\cdot(0.92)}_{\text{trees from last year}} + \underbrace{100\,(0.92)\cdot(0.92)}_{\text{trees from two years ago}} + \underbrace{100\,(0.92)^3}_{\text{trees from 3 years ago}} \cdots$$

$$= \sum_{n=0}^{\infty} 100\cdot(0.92)^n = \frac{100}{1-0.92} = \frac{100}{0.08} =$$

$= \underline{\underline{1250 \text{ trees}}}$

Exercise 4. A falling ball upon hitting the ground bounces back to 30% of the height where the fall started. Initially, the ball was dropped from the height 20 m. If the ball keeps bouncing indefinitely, find the overall distance that the ball travels.



$$\text{Overall distance} = 20 + 2\cdot 20\cdot(0.3) + 2\cdot 20\cdot(0.3)^2 + 2\cdot 20(0.3)^3 + \dots$$

$$= 20 + \Big(\underbrace{(2\cdot 20\cdot 0.3)}_{k=\frac{120}{10}=12} + \underbrace{(2\cdot 20\cdot(0.3))}_{k}\underbrace{(0.3)}_{q} + (2\cdot 20\cdot(0.3))\underbrace{(0.3)^2}_{q^2} + \dots\Big)$$

$$= 20 + \sum_{n=0}^{\infty} 12\cdot(0.3)^n = 20 + \frac{12}{1-0.3} =$$

$$= 20 + \frac{12}{0.7} \approx 37.14 \text{ m}$$

Exercise 5. An investment fund has annual interest rate 6.6%, compounded continuosly. We would like to invest certain amount so that three years from now, we may start annual withdrawals \$3000 indefinitely. How much do we need to invest?

1) How much will an investment grow?

If initially one invests I_0 amount of dollars, then we have $\left[\frac{dI}{dt} = 0.066 \cdot I, \quad I(0) = I_0 \right] \rightsquigarrow$

solving initial value problem $\rightsquigarrow$ $\underline{I(t) = I_0 \cdot e^{0.066t}}$

2) If one invests $\underline{I_0}$ dollars, then:

in one year, it becomes $I_0 \cdot e^{0.066}$ dollars,

in two years, it becomes $I_0 \cdot e^{(0.066)\cdot 2}$ dollars,

⋮

in $\underline{n}$ years, it becomes $I_0 \cdot e^{(0.066)\cdot n}$ dollars

3) In other words, if in n years we expect to have y dollars on the account, we need to invest $\frac{y}{e^{(0.066)\cdot n}}$ dollars now.

4) Thus, we need to invest

$$\underbrace{\frac{3000}{e^{(0.066)\cdot 3}}}_{\text{to be withdrawn 3 years from now as 3000}} + \underbrace{\frac{3000}{e^{(0.066)\cdot 4}}}_{\text{to be withdrawn 4 years from now as 3000}} + \frac{3000}{e^{(0.066)5}} + \dots = \sum_{n=0}^{\infty} \overbrace{\left(\frac{3000}{e^{(0.066)\cdot 3}}\right)}^{k} \cdot \overbrace{\left(\frac{1}{e^{0.066}}\right)^n}^{q}$$

$$= \frac{3000\, e^{-(0.066)\cdot 3}}{1 - e^{-0.066}} \approx$$

$$\approx \underline{38534 \text{ dollars}}$$