MA 16020 Lesson 16: Geometric series I

Series. A (number) series is: an expression of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \dots$$

for some sequence of numbers $(a_n)_n = a_0, a_1, a_2, a_3, \dots$

Example:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

$$\sum_{n=1}^{\infty} \frac{3}{n^2} = 3 + \frac{3}{4} + \frac{3}{9} + \frac{3}{16} + \dots$$

$$\sum_{n=0}^{\infty} (n+2) = 2 + 3 + 4 + 5 + \dots$$

A partial sum of a series is: the sum of first several terms in the series.

Example: For the series

$$\sum_{n=0}^{\infty} \frac{3}{n^2+n+2}, = \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{14} + \dots$$

the first partial sum is $S_0 = \frac{3}{2}$,

the second partial sum is $S_1 = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$,

the third partial sum is $S_2 = 3/2 + 3/4 + 3/8 = \frac{12+6+3}{8} = \frac{21}{8}$

...

The sum of a series is: the sum of all its terms.

(Fact: It can be computed as $S = \lim_{n \to \infty} S_n$, where S_n are the partial sums)

A series is called **convergent** if it has finite sum.

A series is called **divergent** if it does not have finite sum.

A geometric series is: a series of the form

$$\sum_{n=0}^{\infty} k \cdot q^n = k + kq + kq^2 + kq^3 + \dots$$

Finding the sum of a geometric series: let $S = \sum_{n=0}^{\infty} k \cdot q^n = k + \cancel{kq} kq + kq^2 \dots$

We have $S = k + kq + kq^2 + kq^3 + \dots \quad / \cdot q$

$$qS = kq + kq^2 + kq^3 + \dots$$

$\rightsquigarrow$ subtracting, $S - qS = (1-q)S = (k + kq + kq^2 + \dots) - (kq + kq^2 + \dots) = k$

$\rightsquigarrow (1-q)S = k \quad \rightsquigarrow \quad \boxed{S = \frac{k}{1-q}}$

The above works when the geometric series is **convergent**, which happens if and only if $|q| < 1$ (ie $-1 < q < 1$)

If $|q| \geq 1$, the geometric series is **divergent**.

Example. Decide whether the series

$$\sum_{n=0}^{\infty} \frac{5 \cdot 4^{n+1}}{7^n}$$

is geometric and convergent, and if it is, find its sum.

- Geometric?

$$\sum_{n=0}^{\infty} \frac{5 \cdot 4^{n+1}}{7^n} = \sum_{n=0}^{\infty} 5 \cdot 4 \cdot \frac{4^n}{7^n} = \sum_{n=0}^{\infty} \underbrace{20}_{k} \cdot \underbrace{\left(\frac{4}{7}\right)^n}_{q^n} \quad \therefore \text{yes, geometric!}$$

- Convergent?

$q = \frac{4}{7}$, $\left|\frac{4}{7}\right| < 1 \Rightarrow$ yes, convergent.

- Sum?

$$S = \frac{20}{1 - \frac{4}{7}} = \frac{20}{\left(\frac{3}{7}\right)} = \frac{140}{3}$$

Exercise 1. Write

$$4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} + \dots$$

in a compact form.

Point is to find (admittedly, guess) the pattern:

$$4 = 4$$

$$-\frac{8}{3} = 4 \cdot \left(-\frac{2}{3}\right)$$

$$\frac{16}{9} = 4 \cdot \left(\frac{4}{9}\right) = 4 \cdot \left(-\frac{2}{3}\right)^2$$

$$-\frac{32}{27} = 4 \cdot \left(-\frac{8}{27}\right) = 4 \cdot \left(-\frac{2}{3}\right)^3$$

$$\Rightarrow 4 - \frac{8}{3} + \frac{16}{9} - \frac{32}{27} + \dots = \sum_{n=0}^{\infty} 4 \cdot \left(-\frac{2}{3}\right)^n$$

(aside: geometric series, convergent)

Exercise 2. Write the number

$$17.\overline{17} = 17.1717171717\dots$$

in the form of a series.

$$17.\overline{17} = \underbrace{17}_{17} + \underbrace{0.17}_{17 \cdot \frac{1}{100}} + \underbrace{0.0017}_{17 \cdot \frac{1}{10000} = 17 \cdot \left(\frac{1}{100}\right)^2} + \underbrace{0.000017}_{17 \cdot \frac{1}{1000000} = 17 \cdot \left(\frac{1}{100}\right)^3} + \dots$$

$$= \sum_{n=0}^{\infty} 17 \cdot \left(\frac{1}{100}\right)^n$$

(geometric series, convergent)

Exercise 3. Compute

$$\sum_{n=0}^{\infty}\left(\frac{3}{4^n}-\frac{6}{5^n}\right).$$

$$\sum_{n=0}^{\infty}\left(\frac{3}{4^n}-\frac{6}{5^n}\right) = \left(\cancel{3}\,3-6\right)+\left(\frac{3}{4}-\frac{6}{5}\right)+\left(\frac{3}{4^2}-\frac{6}{5^2}\right)+\dots$$

$$=\left(3+\frac{3}{4}+\frac{3}{4^2}+\dots\right)-\left(6+\frac{6}{5}+\frac{6}{5^2}+\dots\right)$$

$$=\underbrace{\sum_{n=0}^{\infty}3\cdot\left(\frac{1}{4}\right)^n}-\underbrace{\sum_{n=0}^{\infty}6\cdot\left(\frac{1}{5}\right)^n}$$

geometric series

$$=\frac{3}{1-\frac{1}{4}}-\frac{6}{1-\frac{1}{5}}=\frac{3}{\left(\frac{3}{4}\right)}-\frac{6}{\left(\frac{4}{5}\right)}=\frac{12}{3}-\frac{30}{4}=$$

$$=-\frac{7}{2}$$

Exercise 4. Approximate the sum

$$\sum_{n=0}^{\infty}\frac{e^{-2n}}{4}$$

to 4 decimal places.

$$S=\sum_{n=0}^{\infty}\frac{e^{-2n}}{4}=\sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{4}\right)}_{a}\cdot\underbrace{\left(e^{-2}\right)^n}_{q}$$

$$=\frac{\frac{1}{4}}{1-e^{-2}}=\frac{1}{4-4e^{-2}}\approx 0.2891$$