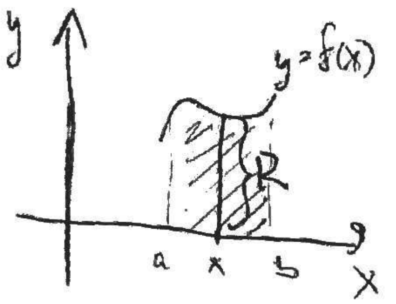
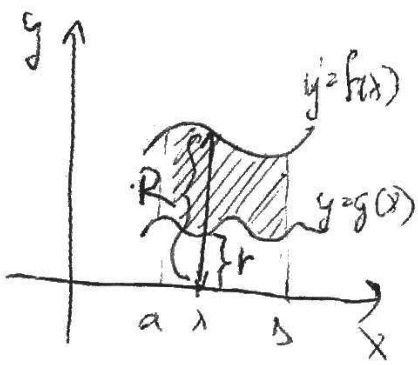
MA 16020 Lesson 14: Volume of solids of revolution III

Recall: Computing volumes of solids of revolution using
the disk method:



$$\text{Volume} = \int_a^b \pi R^2 \, dx \left(= \int_a^b \pi \cdot f(x)^2 \, dx \right)$$

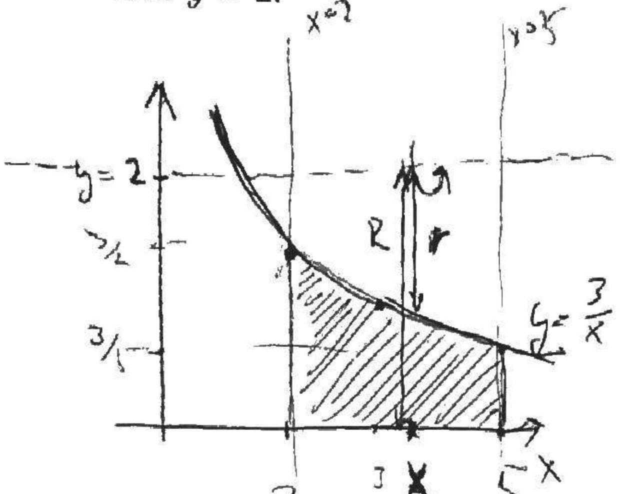
the washer method:



$$\text{Volume} = \int_a^b \pi \left(R^2 - r^2\right) dx \left(= \int_a^b \pi \left(f(x)^2 - g(x)^2\right) dx \right)$$

So far we have considered only rotations with respect to the $x-$ or $y-$axis. Today we consider more general axes.

Exercise 1. Compute the volume of the solid obtained by rotating the region enclosed by the curves $y = 3/x$, $x = 2, x = 5$ and $y = 0$ about the line $y = 2$.



$R(x) = 2$ constantly

$r(x) = 2 - \frac{3}{x}$

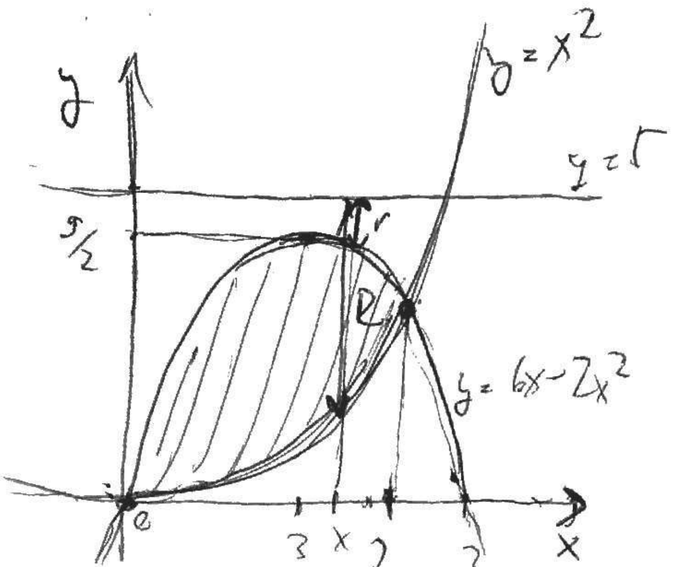
$\rightarrow$ Volume $= \int_2^5 \pi \left(2^2 - \left(2 - \frac{3}{x}\right)^2\right) dx =$

$= \pi \int_2^5 \left(4 - 4 + \frac{12}{x} - \frac{9}{x^2}\right) dx =$

$= \pi \int_2^5 \left(\frac{12}{x} - \frac{9}{x^2}\right) dx = \pi \left[12 \ln(x) + \frac{9}{x}\right]_2^5$

$= \pi \left(12 \ln(5) + \frac{9}{5} - 12 \ln(2) - \frac{9}{2}\right) \approx 26.06$ units

Exercise 2. Compute the volume of the solid obtained by rotating the region enclosed by the curves $y = x^2$ and $y = 6x - 2x^2$ about the line $y = 5$.



$r(x) = 5 - (6x - 2x^2) = 5 - 6x + 2x^2$

$R(x) = 5 - x^2$

$\rightarrow$ Volume $= \int_0^2 \pi\left((5 - x^2)^2 - (5 - 6x + 2x^2)^2\right) dx =$

$= \pi \int_0^2 \left((25 - 10x^2 + x^4) - (25 + 36x^2 + 4x^4 - 60x + 20x^2 - 24x^3)\right)$

$= \pi \int_0^2 25 - 10x^2 + x^4 - 25 - 56x^2 + 60x + 24x^3 - 4x^4$

$= \pi \int_0^2 (-3x^4 + 24x^3 - 66x^2 + 60x) \, dx =$

$= \pi \left[-\frac{3}{5}x^5 + 6x^4 - 22x^3 + 30x^2\right]_0^2$

$= \pi \cdot \frac{104}{5}$

intersections:

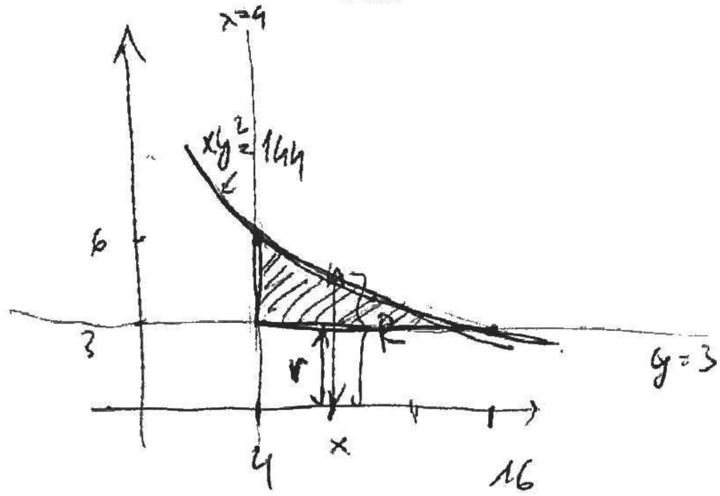
$x^2 = 6x - 2x^2$ $\qquad (2 - x)x = 0$

$6x - 3x^2 = 0$ $\qquad x = 2$ or $x = 0$

$2x - x^2 = 0$ $\qquad (y = 4)$ $\quad (y = 0)$

Exercise 3. Compute the volume of the solid obtained by rotating the region enclosed by the curves $xy^2 = 144, x = 4$ and $y = 3$

(a) about the x-axis:



$r(x) = 3$ constantly

$R(x) =$ the y value for the given x in $xy^2 = 144$

$= \sqrt{\frac{144}{x}}$

$\rightsquigarrow$ Volume $= \int_4^{16} \pi\left(\left(\sqrt{\frac{144}{x}}\right)^2 - 3^2\right) =$

$= \pi \int_4^{16} \left(\frac{144}{x} - 9\right) dx =$

$= \pi \left[144 \ln(x) - 9x\right]_4^{16} =$

$= \pi\left(144 \ln(16) - 9 \cdot 16 - 144 \ln(4) - 9 \cdot 4\right)$

$\approx \underline{287.85 \text{ unit}^3}$

Intersections:

$x = 4,\ xy^2 = 144$

$\to y^2 = \frac{144}{4} = 36$

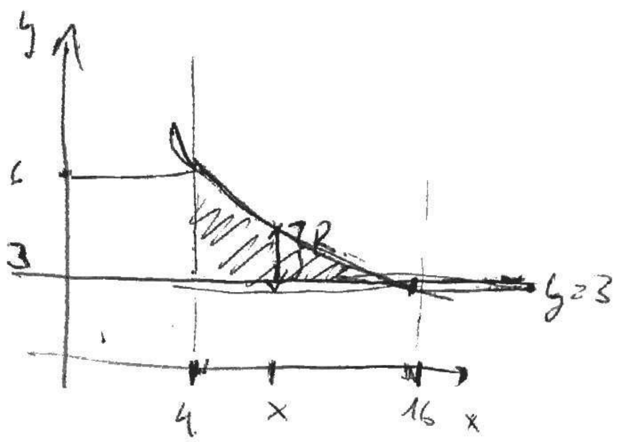
$y = \pm 6$

$y = 3,\ xy^2 = 144$

$x \cdot 9 = 144,$

$x = \frac{144}{9} = 16$

(b) about the line $y = 3$:



$R(x) = \sqrt{\frac{144}{x}} - 3$

$r(x) = 0$ (disk method)

$\to$ Volume $= \int_4^{16} \pi\left(\sqrt{\frac{144}{x}} - 3\right)^2 dx =$

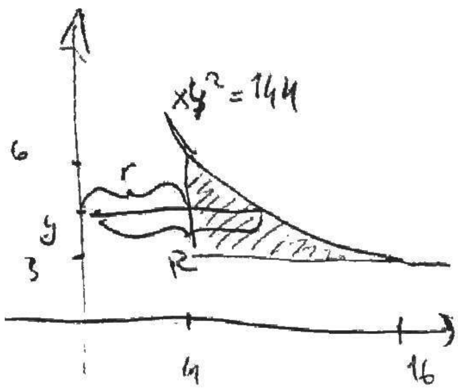
$= \pi \int_4^{16} \left(12x^{-\frac{1}{2}} - 3\right)^2 dx = \pi \int_4^{16} \left(\frac{144}{x} - 72x^{-\frac{1}{2}} + 9\right) dx =$

$= \pi\left[144 \ln(x) - 144x^{\frac{1}{2}} + 9x\right]_4^{16} = \pi\left(144 \ln(16) - 144 \cdot 4 + 9 \cdot 16 - 144 \ln(4) + 144 \cdot 2 - 9 \cdot 4\right)$

$\approx \underline{61.66 \text{ unit}^3}$

Exercise 3 (cont.). Compute the volume of the solid obtained by rotating the region enclosed by the curves $xy^2 = 144$, $x = 4$ and $y = 3$

(c) about the y-axis:



$r(y) = 4$ constantly

$R(y) =$ the x-value for the y in $xy^2 = 144$

$= \frac{144}{y^2}$

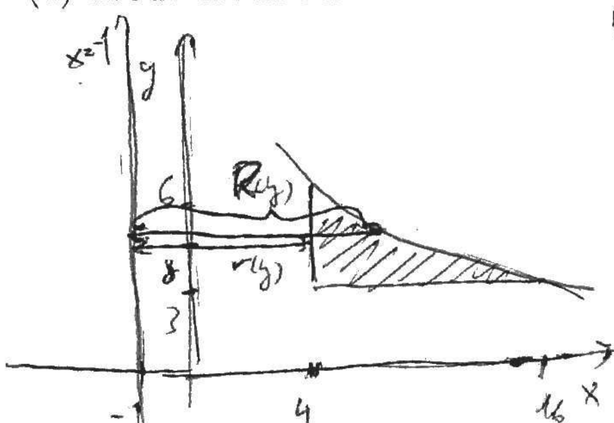
$\to$ Volume $= \int_3^6 \pi\left(\left(\frac{144}{y^2}\right)^2 - 4^2\right) dy =$

$= \pi \int_3^6 \left(20736\, y^{-4} - 16\right) dy =$

$= \pi\left[-\frac{20736}{3} y^{-3} - 16y\right]_3^6 = \pi\left(-\frac{20736}{3\cdot 6^3} - 16\cdot 6 + \frac{20736}{3\cdot 3^3} + 16\cdot 3\right)$

$= \underline{\underline{176\pi}} \approx \underline{\underline{552.92 \text{ unit}^3}}$

(d) about the line $x = -1$:



Volume $= \int_3^6 \pi\left(\left(\frac{144}{y^2}+1\right)^2 - 5^2\right) dy =$

$= \pi \int_3^6 \left(20736 y^{-4} + 288 y^{-2} + 1 - 25\right) dy =$

$= \pi\left[-\frac{20736}{3} y^{-3} - 288 y^{-1} - 24y\right]_3^6$

$= \pi\left(-\frac{20736}{3\cdot 6^3} - \frac{288}{6} - 24\cdot 6 + \frac{20736}{3\cdot 3^3} + \frac{288}{3} + 24\cdot 3\right) =$

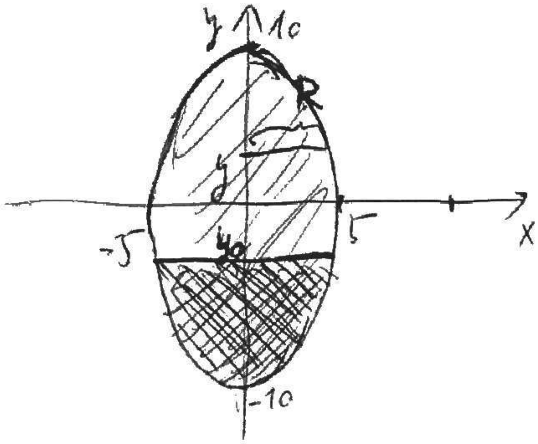
$r(y) = 5$ constantly

$R(y) = (\text{previous } R(y) + 1) =$

$= \frac{144}{y^2} + 1$

$= \underline{\underline{200\pi}} \approx \underline{\underline{628.32 \text{ unit}^3}}$

Exercise 4 (time permitting). The shape of a propane tank is obtained by revolving the interior of $4x^2 + y^2 = 100$ about the y-axis. What is the depth of propane in the tank when it is filled to 1/3 of its capacity?



$R(y)$ = the x-value for the given y on $4x^2+y^2=100$

$= \sqrt{\frac{100-y^2}{4}}$

$\rightarrow$ Volume $= \int_{-10}^{10} \pi \left(\sqrt{\frac{100-y^2}{4}}\right)^2 dy =$

$= \pi \int_{-10}^{10} \cancel{25}\,\frac{100-y^2}{4}\, dy = \pi \int_{-10}^{10} \left(25 - \frac{y^2}{4}\right) dy =$

$= \pi \left[25y - \frac{y^3}{12}\right]_{-10}^{10} = \pi\left(250 - \frac{1000}{12} + 250 - \frac{1000}{12}\right)$

$= \pi\left(500 - \frac{2000}{12}\right) = \cancel{\pi \cdot 1000} = \pi \cdot \frac{1000}{3}$

Intersection with axes:

$x=0 \leadsto 4\cdot 0 + y^2 = 100$
$y^2 = 100$
$\rightarrow y = \pm 10$

$y = 0 \leadsto 4x^2 + 0^2 = 100$
$4x^2 = 100$
$x^2 = 25$
$x = \pm 5$

Want to find y_0 such that

$$\int_{-10}^{y_0} \pi \left(\sqrt{\frac{100-y^2}{4}}\right)^2 dy = \frac{1}{3}\,\text{Volume} = \pi \cdot \frac{1000}{9}$$

$$\int_{-10}^{y_0} \left(25 - \frac{y^2}{4}\right) dy = \frac{1000}{9}$$

⋮

$$\left[25y - \frac{y^3}{12}\right]_{-10}^{y_0} = \frac{1000}{9}$$

get $y_0 \approx$ -16.08 ✗, -18.34 ✗ (not in range $[-10,10]$), -2.26 ✓

$y_0 \approx -2.26$, and the ~~height~~ depth is $(-2.26)-(-10) \approx 7.74$

$25y_0 - \frac{y_0^3}{12} + 250 - \frac{1000}{12} = \frac{1000}{9}$, solve using software

~~$\leadsto$ get $y_0 \approx 18$~~ ≈ 16.0