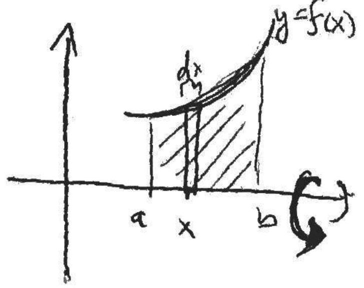
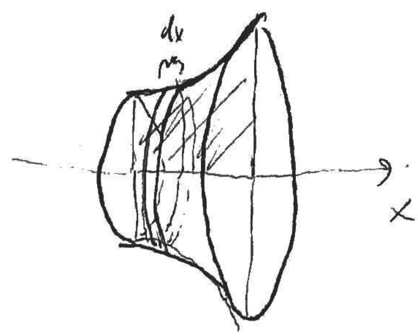
MA 16020 Lesson 13: Volume of solids of revolution II

Last time: Computing volumes of solids of revolution using the disk method.

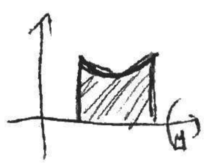


$\rightsquigarrow$

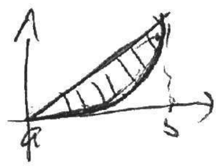


$$\boxed{\text{volume} = \int_a^b \pi \cdot f(x)^2 \, dx}$$

When can the method be applied?
Works when the rotated region is the ~~region~~ full region between a curve and the axis of revolution:



✓

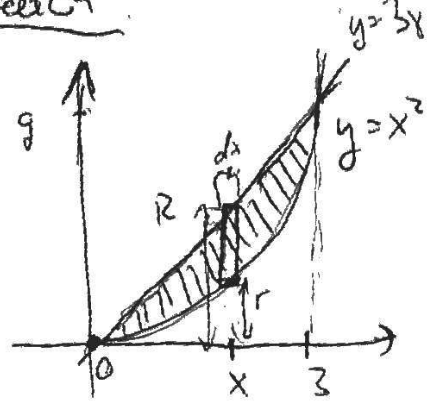


✗

Goal for today: Compute volumes of more general solids of revolution via a *washer method.*

Example: Compute the volume of the solid obtained by rotating the region enclosed by the curves $y = x^2$ and $y = 3x$ about the x-axis.

Sketch

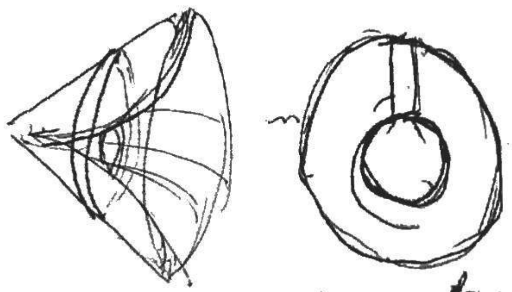


int. points:

$$y=x^2,\ y=3x \rightsquigarrow x^2=3x$$
$$x^2-3x=0$$
$$x(x-3)=0$$
$$\rightarrow \boxed{\begin{matrix} x=0 \\ y=0 \end{matrix}} \text{ or } \boxed{\begin{matrix} x=3 \\ y=3^2=9 \end{matrix}}$$

The disk method is not applicable.

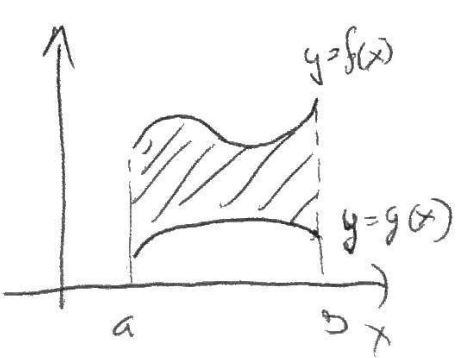
Idea: Instead of thin disks, we consider thin "washers":



$$\text{Volume} = dx \cdot \pi \cdot (R^2-r^2)$$
$$= \pi\left((3x)^2-(x^2)^2\right)dx$$

$$\rightsquigarrow \text{Volume} = \int_0^3 \pi \cdot \left((3x)^2-(x^2)^2\right)dx =$$
$$= \pi\int_0^3 (9x^2-x^4)\,dx$$
$$= \pi\left[3x^3-\frac{x^5}{5}\right]_0^3 =$$
$$= \underline{\underline{\frac{162}{5}\pi \text{ unit}^3}}$$

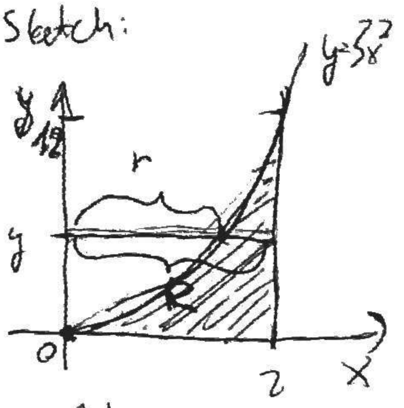
The washer method (for rotating about the x-axis).
Given a region between $y = f(x)$ and $y = g(x)$, $f > g$, over the interval $[a, b]$, the volume is computed as



$$\boxed{\text{Volume} = \int_a^b \pi\left(f(x)^2-g(x)^2\right)dx}$$

Exercise 1. Compute the volume of the solid obtained by rotating the region enclosed by the curves $y = 3x^2$, $x = 2$ and $y = 0$ about the y-axis.

Sketch:



$R = 2$

r = the x-value for the point on the curve

$y = 3x^2$

$\leadsto$ solve for x:

$x^2 = \frac{y}{3}$

$\boxed{x = \sqrt{\frac{y}{3}}}$

int. points:

$y = 3x^2$, $y = 0$: $3x^2 = 0 \Rightarrow x = 0$

$y = 3x^2$, $x = 2$: $y = 3(2^2) = 12$

$$\leadsto \text{Volume} = \int_0^{12} \pi (R^2 - r^2)\, dy =$$

$$= \int_0^{12} \pi \left(4 - \left(\sqrt{\frac{y}{3}}\right)^2\right) dy$$

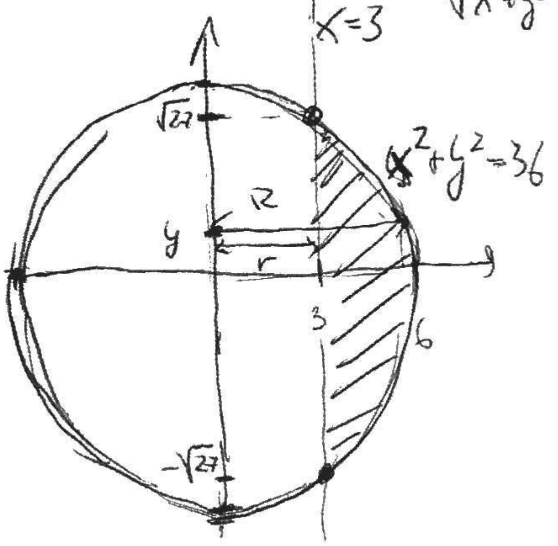
$$= \pi \int_0^{12} \left(4 - \frac{y}{3}\right) dy =$$

$$= \pi \left[4y - \frac{y^2}{6}\right]_0^{12} =$$

$$= 24\pi$$

Exercise 2. Compute the volume of the solid obtained by rotating the region inside $x^2 + y^2 = 36$ and to the right of the line $x = 3$ about the y-axis.

$\sqrt{x^2 + y^2} = 6$



$r = 3$ constantly

$R(y)$ = the corresponding x-value for the given y on the curve

$x^2 + y^2 = 36$

$\leadsto x^2 = 36 - y^2$

$\boxed{x = \sqrt{36 - y^2}}$

Intersection points:

$x^2 + y^2 = 36$, $x = 3$

$\leadsto 3^2 + y^2 = 36$, $y^2 = 36 - 9 = 27$

$\boxed{y = \pm\sqrt{27}}$

$$\text{Volume} = \int_{-\sqrt{27}}^{\sqrt{27}} \pi \left(\left(\sqrt{36 - y^2}\right)^2 - 3^2\right) dy$$

$$= \int_{-\sqrt{27}}^{\sqrt{27}} \pi (36 - y^2 - 9)\, dy =$$

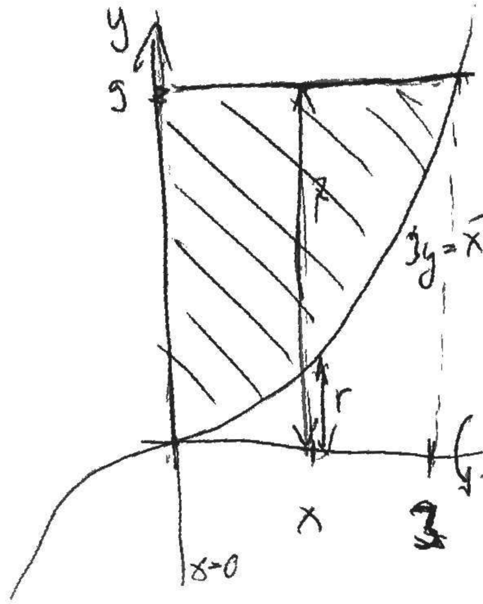
$$= \pi \int_{-\sqrt{27}}^{\sqrt{27}} (27 - y^2)\, dy =$$

$$= \pi \left[27y - \frac{y^3}{3}\right]_{-\sqrt{27}}^{\sqrt{27}} =$$

$$= \pi \left(\left(27 \cdot \sqrt{27} - 9\sqrt{27}\right) - \left(-27\sqrt{27} + 9\sqrt{27}\right)\right) =$$

$$= 36\sqrt{27}\,\pi$$

Exercise 3. Compute the volume of the solid obtained by rotating the region enclosed by the curves $3y = x^3, x = 0$ and $y = 9$
(a) about the x-axis:



$R = 9$ constantly

$r(x)$ = the corresponding y-value on the curve

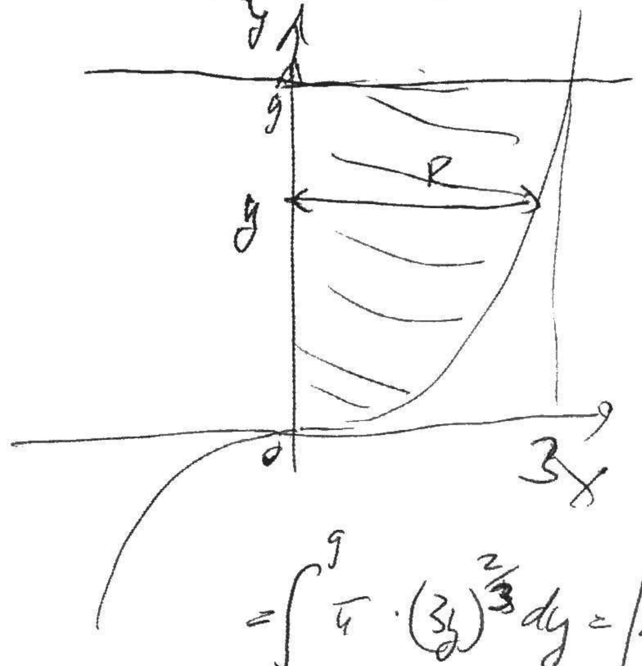
$3y = x^3$

$r(x) = y = \frac{x^3}{3}$

$$\text{Volume} = \int_0^3 \pi\left(9^2 - \left(\frac{x^3}{3}\right)^2\right) dy = \pi\int_0^3 \left(81 - \frac{x^6}{9}\right) dx =$$

$$= \pi\left[81x - \frac{x^7}{63}\right]_0^3 = \pi \cdot \left(81 \cdot 3 - \frac{3^7}{63}\right) = \pi \cdot \frac{1458}{7}$$

(b) about the y-axis:



$R(y)$ = the x-value on $3y = x^3$

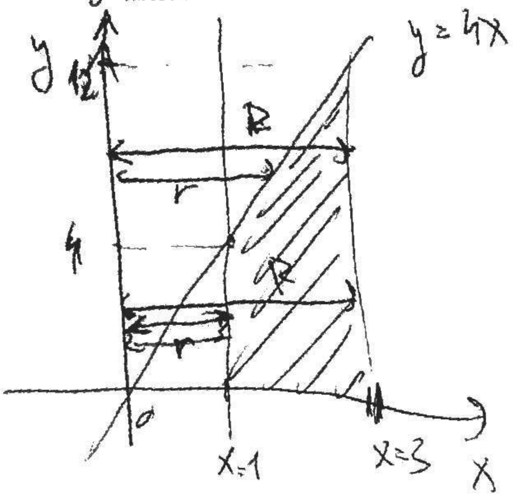
$R(y) = x = \sqrt[3]{3y}$

$$\text{Volume} = \int_0^9 \pi \cdot \left(\sqrt[3]{3y}\right)^2 dy =$$

$$= \int_0^9 \pi \cdot (3y)^{\frac{2}{3}} dy = \left| \begin{matrix} u = 3y \\ du = 3dy \end{matrix} \right| = \frac{\pi}{3}\int_0^3 u^{\frac{2}{3}} du = \frac{\pi}{3}\left[\frac{3}{5} u^{\frac{5}{3}}\right]_0^3$$

$$= \frac{\pi}{5} \cdot 3^{\frac{5}{3}}$$

Exercise 4. Compute the volume of the solid obtained by rotating the region enclosed by the lines $y = 4x, x = 1, x = 3$ and $y = 0$, about the y-axis.



For y's from 0 to 4:

R = 3 constantly
r = 1 constantly

For y's from 4 to 12:

R = 3 constantly
r = the x-value for the given y at y = 4x

$\rightsquigarrow x = \frac{y}{4}$

$r = \frac{y}{4}$

$$\text{Volume} = \int_0^4 \pi(3^2 - 1^2)\,dy + \int_4^{12} \pi\left(3^2 - \left(\frac{y}{4}\right)^2\right) dy$$

$$= \pi\int_0^4 9\,dy + \pi\int_4^{12}\left(9 - \frac{y^2}{16}\right) dy = \pi\Big[9y\Big]_0^4 + \pi\left[9y - \frac{y^3}{48}\right]_4^{12}$$

$$= \pi \cdot 36 + \pi\left(\underbrace{\left(108 - \frac{12^3}{48}\right)}_{72} - \underbrace{\left(36 - \frac{4^3}{48}\right)}_{104/3}\right)$$

$$= \pi\left(36 + \frac{112}{3}\right) = \frac{220}{3}\pi$$