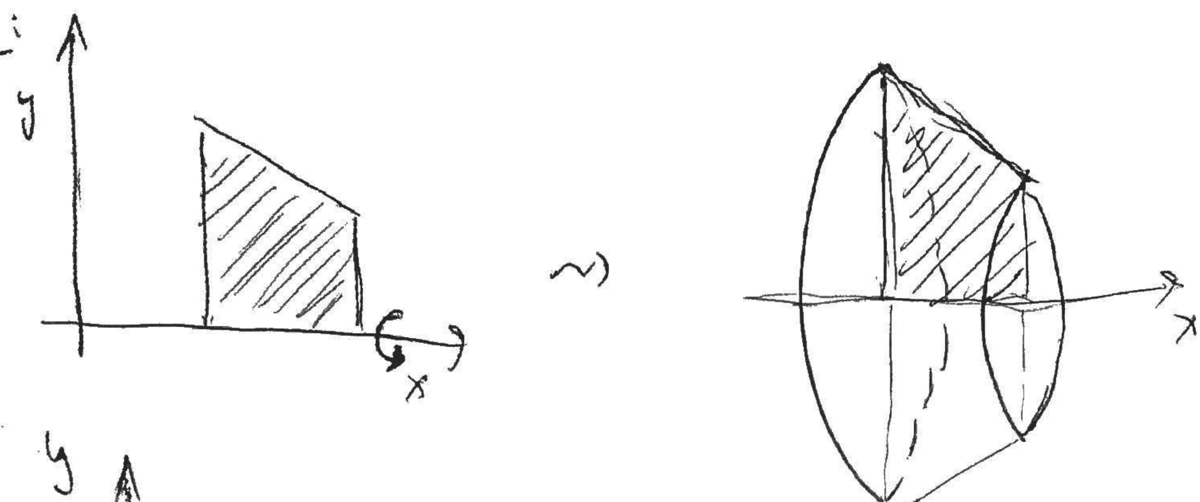
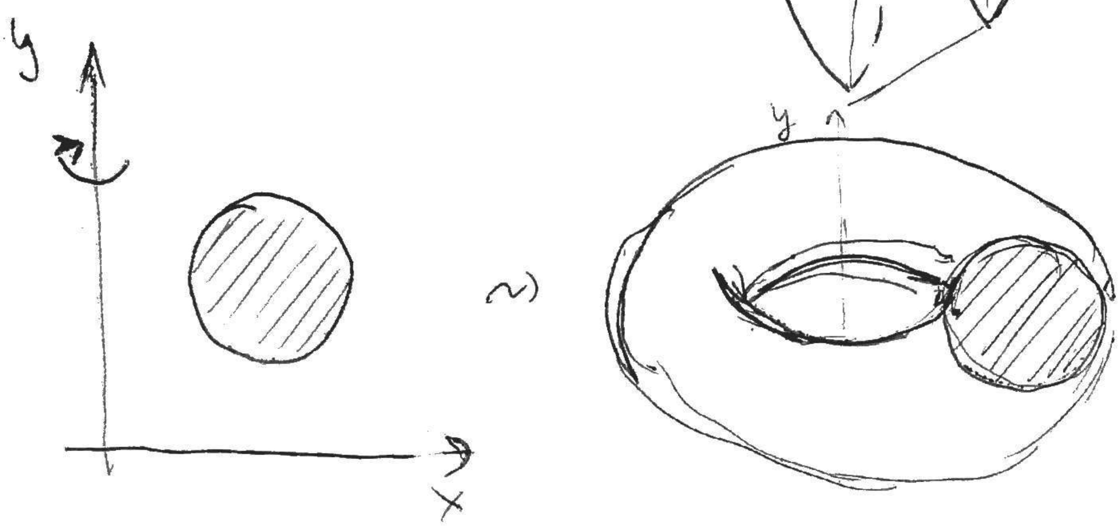
MA 16020 Lesson 12: Volume of solids of revolution I

A solid of revolution is: a solid obtained by rotating a region in a plane around an axis:

Ex:

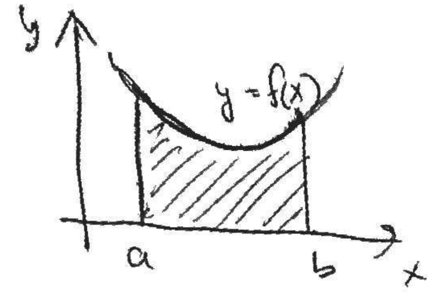


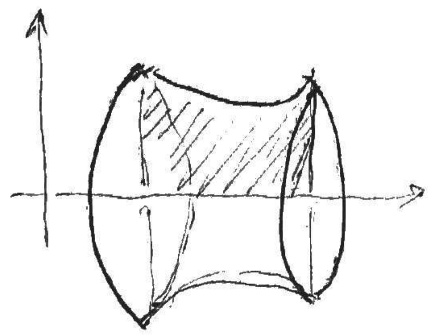


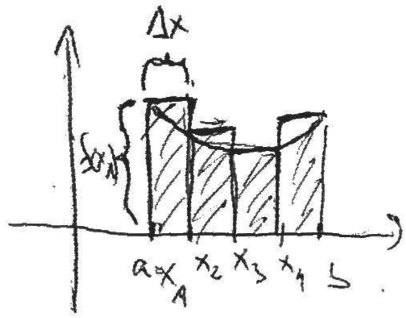
Goal for today: Compute the volume of solids of revolution via a *disk method.*

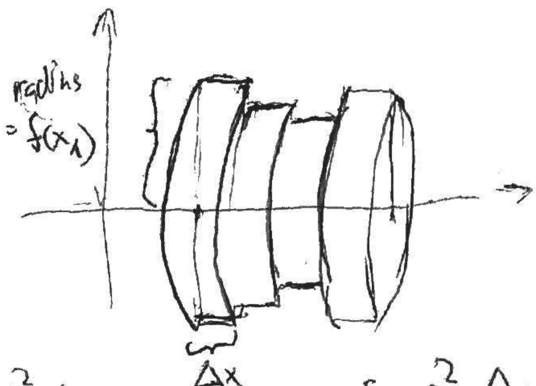
The disk method (for rotating about the x-axis).

Idea: Approximate the volume of the solid by thin disks:









Volume of one disk = $\pi \cdot (\text{radius})^2 \cdot (\text{thickness}) = \pi \cdot f(x_i)^2 \cdot \Delta x$

Volume of the solid $\approx \sum_{i=1}^{n} \pi \cdot f(x_i)^2 \cdot \Delta x$

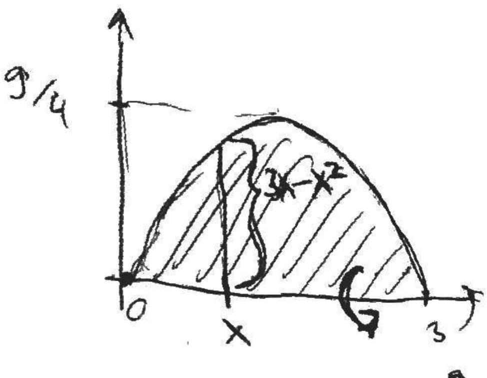
As Δx gets smaller and smaller, the approximation gets better and better.

In the limit of this process, Δx becomes dx and $\sum$ becomes $\int$. So we obtain:

$$\boxed{\text{Volume of the solid} = \int_a^b \pi \cdot f(x)^2 \, dx}$$

Exercise 1. Compute the volume of the solid obtained by rotating the region enclosed by the curve $y = 3x - x^2$ and the x-axis about the x-axis.

Sketch:



int. points with the x-axis:

$3x - x^2 = 0$

$x(3-x) = 0$

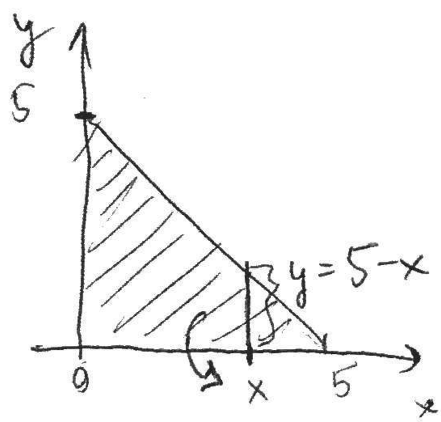
$x = 0$ or $x = 3$

$$\text{Volume} = \int_0^3 \pi \cdot (3x-x^2)^2\,dx = \int_0^3 \pi\,(9x^2 - 6x^3 + x^4)\,dx$$

$$= \pi \cdot \left[\frac{\cancel{9}}{\cancel{3}} 3x^3 - \frac{3}{2}x^4 + \frac{x^5}{5} \right]_0^3 = \pi\left(3\cdot 3^3 - \frac{3}{2}\cdot 3^4 + \frac{3^5}{5}\right) =$$

$$= \pi\, 81/10 = 8.1\pi$$

Exercise 2. Compute the volume of the solid obtained by rotating the region enclosed by the lines $x + y = 5$, $y = 0$ and $x = 0$ about the x-axis.



Intersections of lines

$x+y=5,\ y=0:\ x=5$

$x+y=5,\ x=0:\ y=5$

$$\rightarrow \text{Volume} = \int_0^5 \pi \cdot (5-x)^2\,dx =$$

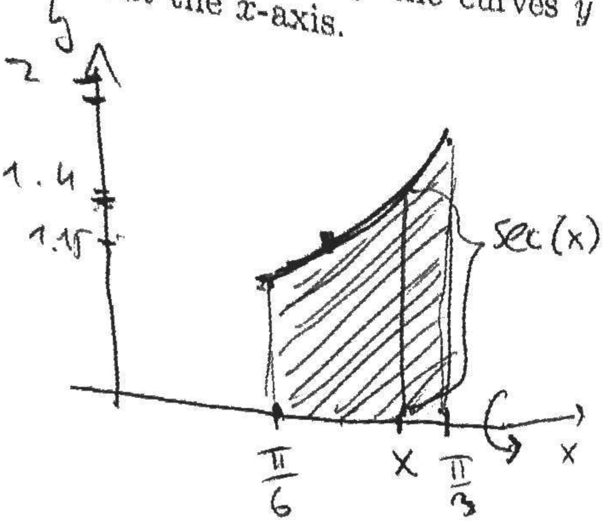
$$= \int_0^5 \pi\,(25 - 10x + x^2)\,dx =$$

$$= \pi \left[25x - 5x^2 + \frac{x^3}{3} \right]_0^5 =$$

$$3 = \pi \cdot \left(125 - 125 + \frac{125}{3}\right) =$$

$$= \frac{125}{3}\pi$$

Exercise 3. Compute the volume of the solid obtained by rotating the region enclosed by the curves $y = \sec(x), y = 0, x = \pi/6$ and $x = \pi/3$ about the x-axis.



$$\text{Volume} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \pi \cdot (\sec(x))^2 \, dx =$$

$$= \pi \cdot \left[\tan(x)\right]_{\pi/6}^{\pi/3} = 4$$

$$= \pi\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) \; (\approx 3.63)$$

~~$y = \sec(x)$~~

~~x | y~~

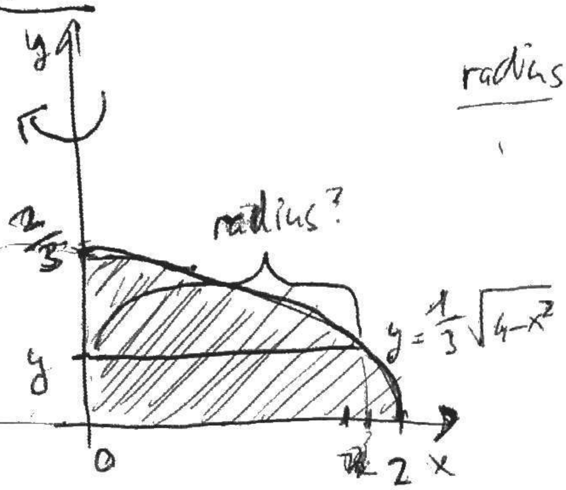
~~π/6~~

~~π/3 | 2~~

$y = \sec(x)$

x	y
$\pi/6$	$\frac{2}{\sqrt{3}} \approx 1.15$
$\pi/4$	$2/\sqrt{2} = \sqrt{2} \approx 1.4$
$\pi/3$	2

Exercise 4. Compute the volume of the solid obtained by rotating the region enclosed by the curves $y = \frac{1}{3}\sqrt{4 - x^2}, y = 0$ and $x = 0$ about the y-axis.



radius: $y = \frac{1}{3}\sqrt{4 - x^2}$

$9y^2 = 4 - x^2$

$x^2 = 4 - 9y^2$

$x = \sqrt{4 - 9y^2}$

$$\rightsquigarrow \text{Volume} = \int_0^{2/3} \pi \cdot \left(\sqrt{4 - 9y^2}\right)^2 dy =$$

$$= \int_0^{2/3} \pi \cdot (4 - 9y^2)\, dy =$$

$$= \pi \left[4y - 3y^3\right]_0^{2/3} = \pi \cdot \frac{16}{9}$$

intersections:

$y = \frac{1}{3}\sqrt{4 - x^2}$, $x = 0$: $y = \frac{2}{3}$

$y = \frac{1}{3}\sqrt{4 - x^2}$, $y = 0$:

$\sqrt{4 - x^2} = 0$

$4 = x^2 \quad x = \pm 2$