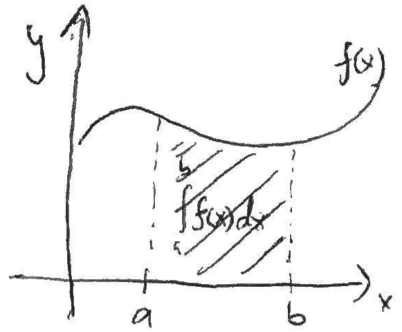
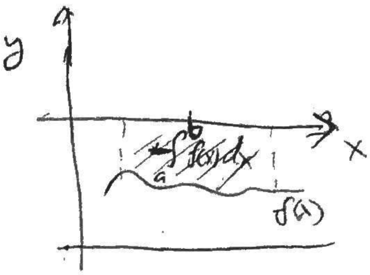
MA 16020 Lesson 11: Areas between curves

Recall: The geometric meaning of the integral $\int_a^b f(x)\mathrm{d}x$ is:

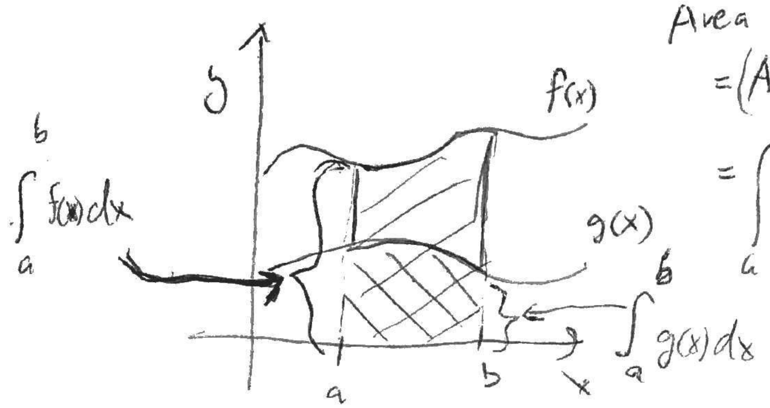
$\int_a^b f(x)dx$ = area of the region below $f(x)$ and above the x-axis over the interval $[a,b]$





Suppose that we have two functions $f(x), g(x)$ such that $f(x) > g(x)$ on a given interval $[a, b]$.

Question 1: How to compute the area between the graphs of $f(x)$ and $g(x)$ over the interval $[a, b]$?

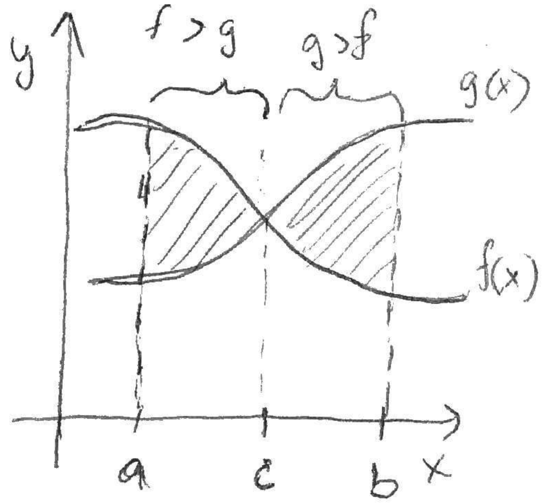


Area between f and g =

= (Area below f) − (Area below g)

$= \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx$

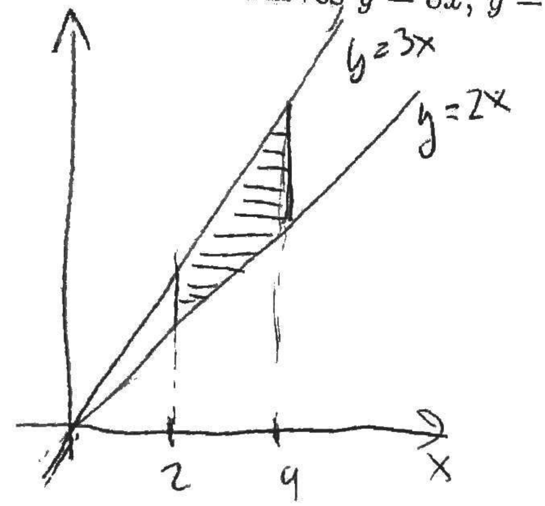
Question 2: What if the functions $f(x), g(x)$ "cross each other"?



Need to find c where $f(c) = g(c)$

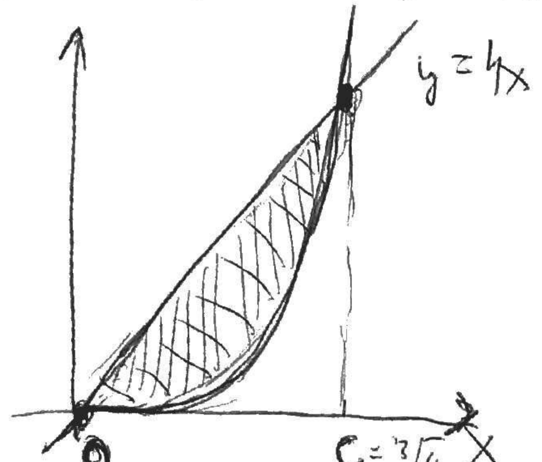
Then Area $= \int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx$

Exercise 1. Sketch the region and set up an integral computing the area
(a) between the curves $y = 3x$, $y = 2x$ over the interval $2 \le x \le 4$:



$$\text{Area} = \int_2^4 (3x - 2x)\,dx = \int_2^4 x\,dx$$

(b) enclosed by the curves $y = x^4$, $y = 4x$:

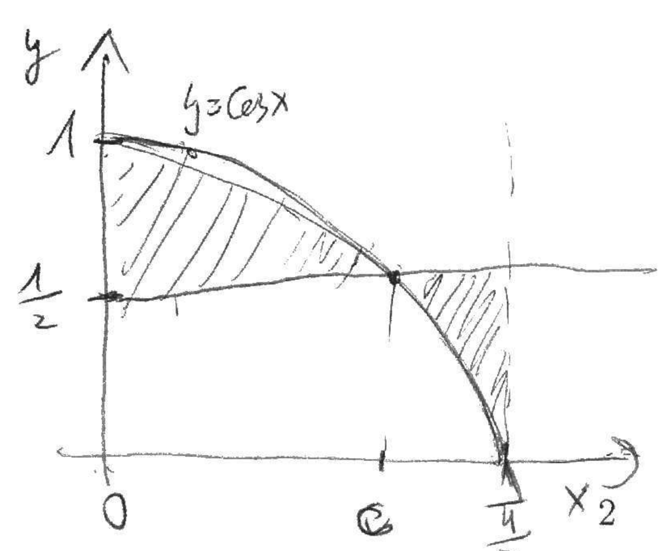


compute c's

$$x^4 = 4x$$
$$x^4 - 4x = 0$$
$$x(x^3-4) = 0$$
$$x = 0 \quad \text{or} \quad x^3 - 4 = 0$$
$$x^3 = 4 \quad \boxed{x = \sqrt[3]{4}} =: c$$

$$\text{Area} = \int_0^{\sqrt[3]{4}} (4x - x^4)\,dx$$

(c) between the curves $y = \cos(x)$, $y = 1/2$ over the interval $0 \le x \le \pi/2$:

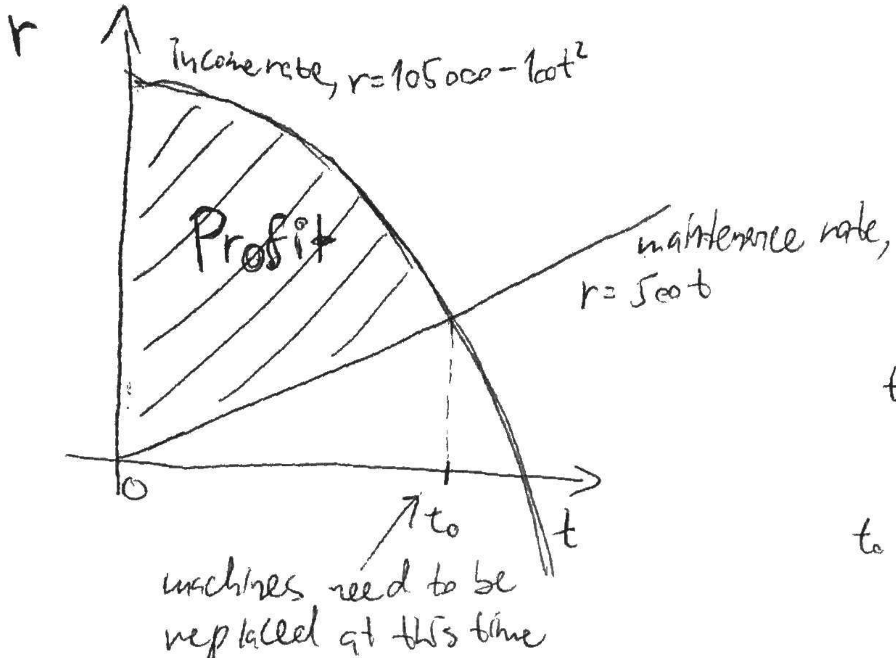


compute c's

$$\cos x = \frac{1}{2}$$
$$\boxed{x = \frac{\pi}{3} = c}$$

$$\text{Area} = \int_0^{\frac{\pi}{3}} \left(\cos(x) - \frac{1}{2}\right)dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{2} - \cos(x)\right)dx$$

Exercise 2. A company installs in its factory new machines that are expected to provide income at the rate of $105000 - 100t^2$ dollars per year, where t is the number of years since installation. On the other hand, the maintenance cost for the machines is $500t$ dollars per year. What is the overall profit of the company from these machines before they need to be replaced?



Compute t_0:

$105000 - 100t^2 = 500t$

$-100t^2 - 500t + 105000 = 0 \quad / : (-100)$

$t^2 + 5t - 1050 = 0$

$t_{1,2} = \dfrac{-5 \pm \sqrt{25 + 4 \cdot 1050}}{2} = \begin{cases} -35 \\ 30 \end{cases}$

t_0 is positive $\Rightarrow$ $t_0 = 30$

$\Rightarrow$ Profit $= \displaystyle\int_0^{30} \left((105000 - 100t^2) - 500t \right) dt =$

$= \left[105000t - \frac{100}{3}t^3 - 250t^2 \right]_0^{30} =$

$= 3\,825\,000$ dollars

Exercise 3. A company expects growth of its profits at the continuous rate between 5% and 15%. The company's profit within the past year was 21 million dollars. Find the positive cumulative difference in predicted total profits over the next 3 years.

~~P(t) = profit rate at years from now~~

$P_{min}(t)$ = minimal profit rate t years from now

$P_{max}(t)$ = maximal — " —

1) $\dfrac{dP_{min}}{dt} = 0.05\, P_{min}$

$\displaystyle\int \frac{dP_{min}}{P_{min}} = \int 0.05\, dt$

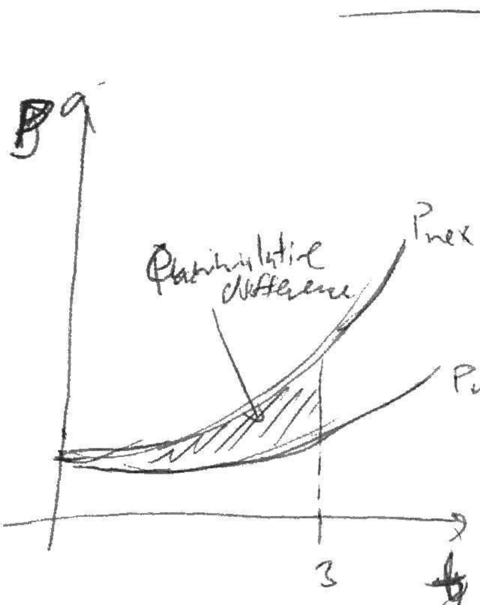
$\log(P_{min}) = 0.05t + C$

$P_{min} = C \cdot e^{0.05t}$, using $P(0) = 21$,

got $\underline{P_{min} = 21 \cdot e^{0.05t}}$

2) Similarly,

$\underline{P_{max} = 21 \cdot e^{0.15t}}$



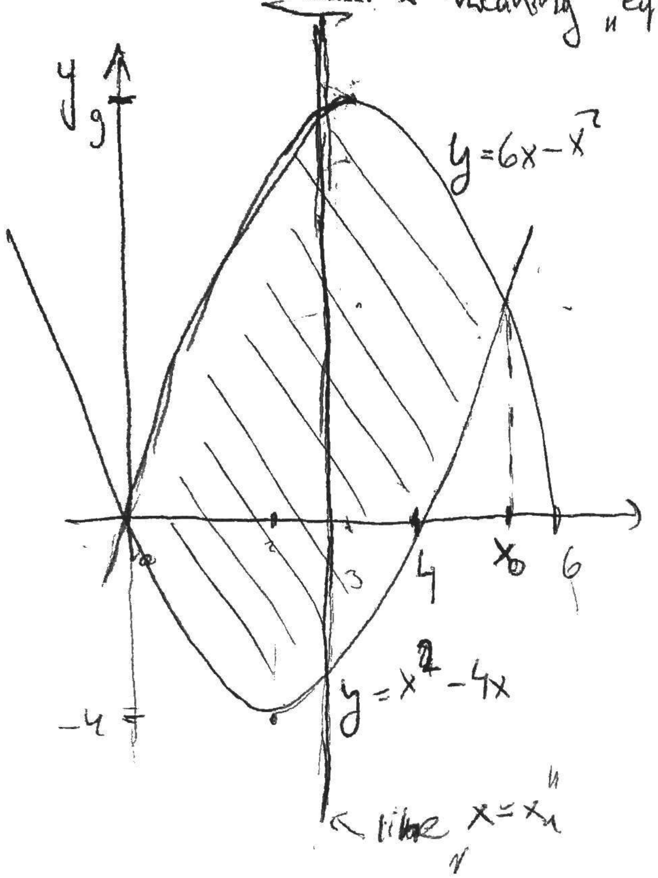
Cumulative difference

$$= \int_0^3 \left(21 e^{0.15t} - 21 e^{0.05t}\right) dt$$

$$= \left[\frac{21}{0.15} e^{0.15t} - \frac{21}{0.05} e^{0.05t}\right]_0^3$$

$$\approx 11.59 \text{ mil. dollars}$$

Exercise 4 (if time permits). Find the equation of the vertical line that divides the area of the region enclosed by the curves $y = 6x - x^2$, $y = x^2 - 4x$ in half. meaning „equal area on both sides"



Compute x_0:

$$6x - x^2 = x^2 - 4x$$
$$2x^2 - 6x = 0$$
$$x^2 - 5x = 0$$
$$x(x-5) = 0 \quad \ldots \quad x=0 \text{ or } x=5$$

$$\text{Area} = \int_0^5 \left((6x-x^2) - (x^2-4x)\right) dx =$$
$$= \int_0^5 (10x - 2x^2)\, dx = \left[5x^2 - \frac{2}{3}x^3\right]_0^5$$
$$= \frac{125}{3}$$

Want: x_1 such that $\int_0^{x_1} \left((6x-x^2)-(x^2-4x)\right)dx = \frac{125}{6}$

$$\int_0^{x_1} (10x - 2x^2)\,dx = \frac{125}{6}$$
$$\left[5x^2 - \frac{2}{3}x^3\right]_0^{x_1} = \frac{125}{6}$$
$$5x_1^2 - \frac{2}{3}x_1^3 = \frac{125}{6}$$

$$x_1^3 - \frac{15}{2}x^2 + \frac{125}{4} = 0$$

(solve eq. with software)

$$x_1 = \frac{5}{2},\ \frac{5}{2} - \frac{5\sqrt{3}}{2} \text{ or } \frac{5}{2} + \frac{5\sqrt{3}}{2}$$

$\rightarrow \boxed{x_1 = \frac{5}{2}}$ is the only one lying between 0 and 5

$\rightarrow$ The line is $x = \frac{5}{2}$