MA 16020 Lesson 10: First-order linear differential equations II

Recap (method of integrating factors, abridged).

$$y' + P(x)y = Q(x)$$

$$u(x) = e^{\int P(x)dx}$$

$$u(x)\cdot y = \int Q(x)u(x)dx + C$$

$$\boxed{y = \frac{\int Q(x)u(x)dx + C}{u(x)}}$$

Exercise 1. Find the general solution to the diff. equation

$$3t^2y' - 2ty = t\sqrt{t}, \quad t > 0 \qquad /:3t^2$$

$$y' \underbrace{- \frac{2}{3}\cdot\frac{1}{t}}_{P(t)} y = \underbrace{\frac{1}{3}t^{-\frac{1}{2}}}_{Q(t)}$$

$$u(t) = e^{\int\left(-\frac{2}{3}\cdot\frac{1}{t}\right)dt} = e^{-\frac{2}{3}\ln|t|} \overset{\boxed{t>0}}{=} e^{-\frac{2}{3}\ln(t)} = \left(e^{\ln(t)}\right)^{-\frac{2}{3}} = t^{-\frac{2}{3}}$$

$$t^{-\frac{2}{3}}y = \int \frac{1}{3}t^{-\frac{1}{2}}\cdot t^{-\frac{2}{3}}dt = \int \frac{1}{3}t^{-\frac{3+4}{6}}dt = \int \frac{1}{3}t^{-\frac{7}{6}}dt = -2t^{-\frac{1}{6}} + C$$

$$\rightsquigarrow \quad y = t^{2/3}\left(-2t^{-\frac{1}{6}} + C\right) = -2\cdot t^{-\frac{1}{6}+\frac{2}{3}} + C\cdot t^{\frac{2}{3}} =$$

$$= -2t^{\frac{1}{2}} + Ct^{\frac{2}{3}}$$

Exercise 2. A chemical substance dissolves in water at the rate proportional to the amount of undissolved substance. Initially, 50 grams of the substance was put in water. After 10 minutes, exactly 25 grams of the substance dissolved. How much of the substance was dissolved after additional 5 minutes?

$A(t)$ = amount of dissolved substance after t min

$A(0)=0$; $A(10)=25$ Want: $A(15)$

$\frac{dA}{dt} = k\underbrace{(50-A)}_{\text{amount of undissolved substance}}$

$\rightsquigarrow \frac{dA}{dt} + kA = 50k$

$u(t) = e^{\int k\,dt} = e^{kt}$

$e^{kt}A = \int 50k \cdot e^{kt}\,dt =$

$= 50e^{kt} + C$

$A = 50 + C \cdot e^{-kt}$

$A = 50 + C \cdot D^t$

OR: $\frac{1}{50-A}\frac{dA}{dt} = k$

$\int \frac{dA}{50-A} = \int k\,dt$

$-\ln(50-A) = kt + C$

$\ln(50-A) = -kt + C$

$50 - A = \cancel{\ }\pm e^{C} \cdot e^{-kt}$

$A = 50 \mp e^{C} \cdot e^{-kt} = 50 + C \cdot D^t$

$A(0)=0$

$0 = 50 + C \cdot 1 \Rightarrow C = -50$

$A(10)=25$:

$25 = 50 - 50 \cdot D^{10}$

$-25 = -50\,D^{10}$

$D^{10} = \frac{1}{2}$

$D = \left(\frac{1}{2}\right)^{\frac{1}{10}}$

$\Rightarrow A(t) = 50 - 50 \cdot \left(\frac{1}{2}\right)^{\frac{t}{10}}$

$A(15) = 50 - 50 \cdot \left(\frac{1}{2}\right)^{\frac{15}{10}} \approx 32.322$ grams

Exercise 3. A company's net worth is initially 10 million dollars. The company expects a 20% yearly growth and, moreover, it expects an extra income from 30% of the growing market estimated at $100e^{0.4t}$ dollars per year, where t is the number of years from now. What is the expected value of the company after 5 years?

$V(t)$ = value of the company after t years.

$V(0) = 10$ $\qquad$ want: $V(5)$

$$\frac{dV}{dt} = \underbrace{0.2V}_{\text{"20\% yearly growth"}} + \underbrace{(0.3)(100e^{0.4t})}_{\text{"30\% from growing market"}}$$

$$\frac{dV}{dt} - 0.2V = 30e^{0.4t}$$

$$u(t) = e^{\int(-0.2)dt} = e^{-0.2t}$$

$$e^{-0.2t}V = \int 30 \cdot e^{0.2t}\,dt = 30 \cdot 5 \cdot e^{0.2t} + C = 150e^{0.2t} + C$$

$$V = 150 \cdot e^{0.4t} + C \cdot e^{0.2t}$$

$V(0) = 10$:

$$10 = 150 \cdot e^0 + C \cdot e^0 = 150 + C$$

$$\Rightarrow C = 10 - 150 = -140$$

$$\rightarrow V(t) = 150 \cdot e^{0.4t} - 140e^{0.2t}$$

$$V(5) = 150 \cdot e^2 - 140 \cdot e^1 \approx 727.80 \text{ mil. dollars}$$

Exercise 4. A 15000 cubic foot room has initially a radon level 810 picocuries per cubic foot. A ventilation system brings in 500 cubic feet of air per hour, and the same amount of mixed air leaves the room. The incoming air has radon level 5 picocuries per cubic foot. How long will it take for the room to attain the safe radon level of 100 picocuries per cubic foot?

$A(t)$ = amount of radon in the room [pCi]

$A(0) = (810\ pCi/ft^3)\times(15000\ ft^3) = 12150000\ pCi$

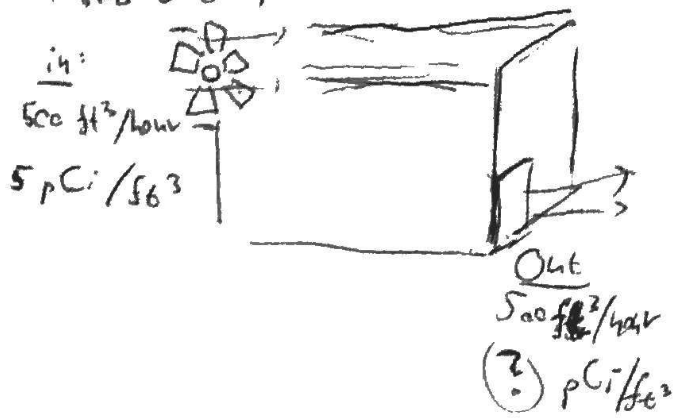
Want: t such that radon level $= 100\ pCi/ft^3$,

ie t st $A(t) = 100 * 15000 = 1500000\ pCi$

$\frac{dA}{dt}$ = rate in − rate out =

$= 5\cdot 500 - 500\cdot\frac{A(t)}{15000}$

$$\boxed{\frac{dA}{dt} = 2500 - \frac{A}{30}}$$



$\frac{dA}{dt} + \frac{A}{30} = 2500$

$u(t) = e^{\int \frac{1}{30}dt} = e^{\frac{t}{30}}$

$e^{\frac{t}{30}}\cdot A = \int 2500\cdot e^{t/30}\,dt$

$= 2500\cdot 30\cdot e^{t/30} + C$

$$A = 75000 + C\cdot e^{-\frac{t}{30}}$$

$A(0) = 12150000$

$12150000 = 75000 + C\cdot e^{0}$

$\rightsquigarrow C = 12150000 - 75000 = 12075000$

$$A(t) = 75000 + 12075000\cdot e^{-\frac{t}{30}}$$

Finally, solve $A(t) = 1500000$.

$1500000 = 75000 + 12075000\,e^{-\frac{t}{30}}$

$e^{-\frac{t}{30}} = \frac{1425000}{12075000}$

$-\frac{t}{30} = \ln\left(\frac{1425000}{12075000}\right)$

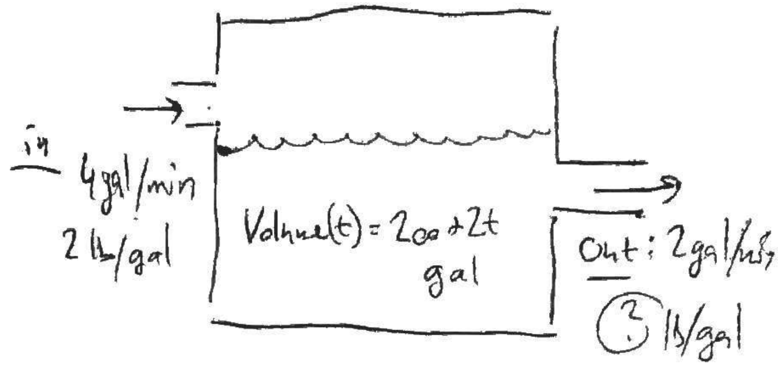
$t = -30\cdot\ln\left(\frac{1425000}{12075000}\right)$

≈ 64.11 hours

Exercise 5 (if time permits). A 600-gallon tank initially contains 200 gallons of brine containing 0.5 pounds of salt per gallon. Brine containing 2 pounds of salt per gallon flows into the tank at the rate of 4 gallons per minute, and the well-stirred mixture flows out of the tank at the rate of 2 gallons per minute. What is the amount of salt in the tank at the time when the tank is full?

$A(t)$ = amount of salt after t minutes

$A(0) = (0.5 \text{ lb/gal}) \times 200 \text{ gal} = 100 \text{ lb}$



$$\frac{dA}{dt} = \text{rate in} - \text{rate out} =$$

$$\frac{dA}{dt} = 4 \cdot 2 - 2 \cdot \frac{A}{200+2t}$$

$$\boxed{\frac{dA}{dt} = 8 - \frac{A}{100+t}}$$

time when the tank is full:

$$200 + 2t = 600$$

$$t = 200$$

want to know $A(200)$

$$\frac{dA}{dt} + \frac{1}{100+t} A = 8$$

$$u(t) = e^{\int \frac{1}{100+t} dt} = e^{\ln(100+t)} = 100+t$$

$$(100+t)A = \int 8(100+t)\, dt =$$

$$= \int (800 + 8t)\, dt = 800t + 4t^2 + C$$

$$A = \frac{800t + 4t^2 + C}{100+t}$$

$A(0) = 100$;

$$100 = \frac{0 + 0 + C}{100}$$

$$C = 100 \cdot 100 = 10\,000$$

$$\boxed{A(t) = \frac{800t + 4t^2 + 10\,000}{100+t}}$$

$$A(200) = \frac{800 \cdot 200 + 4 \cdot (200)^2 + 10\,000}{300}$$

$= 1100$ pounds of salt