MA 16010 Lesson 9: Product rule

Recal: Computational rules for limits that we know so far are:

1. **Constant rule:** $\frac{d}{dx}[c] = 0$

   + Constant multiple rule: $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$

2. **Power rule:** $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$

3. **Sum, difference rules:** $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

4. **Derivatives of basic functions:**

$\frac{d}{dx}[\sin(x)] = \cos(x)$ $\quad \frac{d}{dx}[\cos(x)] = -\sin(x)$ $\quad \frac{d}{dx}[e^x] = e^x$

Today we add a new one:

5. **Product rule:** Given two functions $f(x), g(x)$, we have

$$\boxed{\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)}$$

Exercise: Compute $h'(x)$ when $h(x) = (3x^2 - 4)(x + 2\sqrt{x})$. $= \underbrace{(3x^2-4)}_{f(x)}\underbrace{(x+2x^{\frac{1}{2}})}_{g(x)}$

(a) With the product rule:

$$f'(x) = \underbrace{(6x)}_{f'(x)} \cdot \underbrace{(x+2x^{\frac{1}{2}})}_{g(x)} + \underbrace{(3x^2-4)}_{f(x)}\underbrace{(1+2 \cdot \tfrac{1}{2}x^{-\frac{1}{2}})}_{g'(x)} =$$

$$= 6x^2 + 12x^{3/2} + 3x^2 + 3x^{3/2} - 4 - 4x^{-1/2} =$$

$$= 9x^2 + 15x^{3/2} - 4 - 4x^{-1/2}$$

(b) Without the product rule:

$$h'(x) = \frac{d}{dx}\left[3x^3 - 4x + 6x^{5/2} - 8x^{1/2}\right] = 3 \cdot 3x^2 - 4 + 6 \cdot \tfrac{5}{2}x^{\frac{3}{2}} - 8 \cdot \tfrac{1}{2}x^{-\frac{1}{2}}$$

$$= 9x^2 - 4 + 15x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}$$ (... same result)

Exercise: Compute $y'(\pi)$ when

$$y = \frac{\sin(x)}{x^2} \quad : = x^{-2} \cdot \sin x$$

$$y'(x) = \frac{d}{dx}(x^{-2}) \cdot \sin x + x^{-2} \cdot \frac{d}{dx}(\sin x) =$$

$$= -2x^{-3} \sin x + x^{-2} (\cancel{1} \cos x) = -\frac{2 \sin x}{x^3} + \frac{\cos x}{x^2}$$

$$y'(\pi) = -\frac{2 \underbrace{\sin(\pi)}_{=0}}{\pi^3} + \frac{\underbrace{\cos(\pi)}_{-1}}{\pi^2} = -\frac{1}{\pi^2}$$

Exercise: Find all x where the graph of the function $f(x)$ has a horizontal tangent line, where

$$f(x) = (x^2 - 2x)e^x \quad .$$

Tangent line is horizontal $\Leftrightarrow$ $f'(x) = 0$

$$f'(x) = \frac{d}{dx}[x^2 - 2x] e^x + (x^2 - 2x) \cdot \frac{d}{dx}[e^x] =$$

$$= (2x - 2) \cdot e^x + (x^2 - 2x) \cdot e^x =$$

$$= (x^2 - 2x + 2x - 2) e^x = (x^2 - 2)e^x$$

$$f'(x) = 0 \Leftrightarrow (x^2 - 2) \underbrace{e^x}_{\text{never } 0!} = 0 \Leftrightarrow x^2 - 2 = 0$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

Exercise: Find the equation of the tangent line to $y = 3x\sin(x)$ at $x = \pi/2$.

1) $\underline{\text{Slope of the tangent line} = y'\left(\frac{\pi}{2}\right)}$:

$y'(x) = 3\cdot\sin(x) + 3x\cdot\cos(x)$

$y'\left(\frac{\pi}{2}\right) = 3\cdot\sin\left(\frac{\pi}{2}\right) + 3\frac{\pi}{2}\cdot\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0}$

$= 3\cdot\sin\left(\frac{\pi}{2}\right) = \underline{3}$

2) tangent line:

$\left(t(x) = f(x_0) + f'(x_0)(x - x_0)\text{"}\right)$

$t(x) = \underbrace{3\cdot\frac{\pi}{2}\cdot\sin\left(\frac{\pi}{2}\right)}_{\text{"}y\left(\frac{\pi}{2}\right)\text{"}} + 3\cdot\left(x - \frac{\pi}{2}\right)$

$= \frac{3\pi}{2} + \underline{\underline{3\left(x - \frac{\pi}{2}\right)}}$

$\underline{\underline{(= 3x)}}$

Exercise: Compute $f'(x)$ when

(a) $f(x) = e^{2x}$:

$f'(x) = \frac{d}{dx}\left[e^x\cdot e^x\right] = \frac{d}{dx}\left[e^x\right]\cdot e^x + e^x\cdot\frac{d}{dx}\left[e^x\right] =$

$= e^x\cdot e^x + e^x\cdot e^x = e^{2x} + e^{2x} = \underline{\underline{2e^{2x}}}$

(b) $f(x) = \sin(2x)$:

$f'(x) = \frac{d}{dx}\left(\sin(2x)\right) = \frac{d}{dx}\left(2\sin(x)\cdot\cos(x)\right) = \frac{d}{dx}\left[2\sin(x)\right]\cdot\cos x +$

$+ 2\sin(x)\cdot\frac{d}{dx}\left[\cos(x)\right] = 2\cos(x)\cdot\cos x + 2\cdot\sin(x)\cdot(-\sin(x))$

$= \underline{\underline{2\left(\cos^2(x) - \sin^2(x)\right)}} \left(= 2\cdot\cos(2x)\right)$

EXTRAS (<u>not needed</u> for homework or class; purely for anyone's interest)

1. Justify the product rule by continuing the limit computation (you may assume that $f(x), g(x)$ are continuous):

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)\cdot g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} =$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} =$$

$= \cdots$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h} =$$

$$= \underbrace{\left(\lim_{h\to 0} g(x+h)\right)}_{=g(x)}\cdot\underbrace{\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)}_{=f'(x)} + \underbrace{\left(\lim_{h\to 0} f(x)\right)}_{=f(x)}\cdot\underbrace{\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)}_{=g'(x)}$$

$$= g(x)\cdot f'(x) + f(x)\cdot g'(x)$$

2. Deduce the quotient rule: using that $g(x)\cdot\frac{f(x)}{g(x)} = f(x)$ and the product rule, deduce the formula for $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right]$.

$$g(x)\cdot\frac{f(x)}{g(x)} = f(x)$$

$$\Rightarrow \frac{d}{dx}\left[g(x)\cdot\frac{f(x)}{g(x)}\right] = f'(x)$$

$$g'(x)\cdot\left(\frac{f(x)}{g(x)}\right) + g(x)\cdot\underbrace{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]}_{\text{need to isolate this}} = f'(x)$$

$$g(x)\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = f'(x) - g'(x)\cdot\frac{f(x)}{g(x)}$$

$$\rightarrow \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{1}{g(x)}\left(f'(x) - g'(x)\cdot\frac{f(x)}{g(x)}\right) =$$

$$= \frac{f'(x)}{g(x)} - \frac{g'(x)f(x)}{(g(x))^2} =$$

$$= \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$