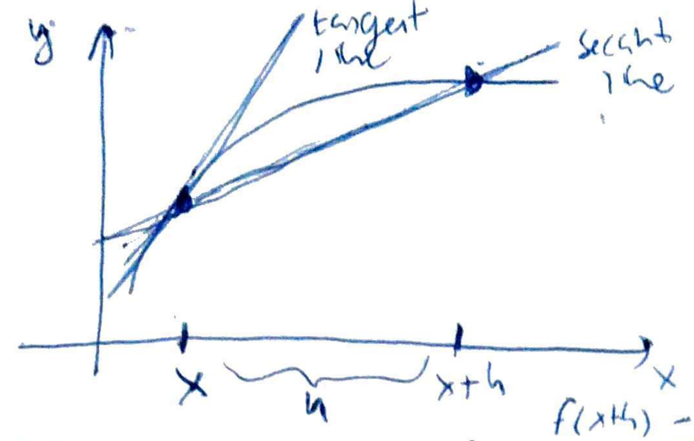
MA 16010 Lesson 8: Instantaneous rate of change

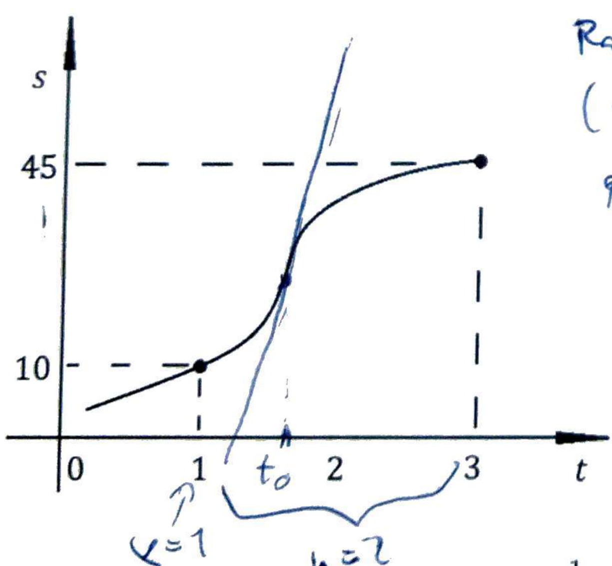
Average vs. instantaneous rate of change. Suppose that $y = f(x)$ is a function, consider some x and some change in x, $\Delta x = h$:



1. The slope of the **secant** line is computed as $\frac{f(x+h) - f(x)}{h}$. Its meaning is the average of rate of change of f from x to x+h (=how fast will f grow from x to $x + h$ on average).

2. The slope of the **tangent** line is computed as $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. Its meaning is the instantaneous rate of change of f at x (=how fast f grows at x, or how fast f grows really close to x).

Example by picture: The following is the graph of distance a car traveled (in m) with respect to time (in s). A radar gun measures the distance at $t = 1$, and then $t = 3$, and estimates the speed. The speed limit is 50 mph ≈ 22.35 m/s. Is the car speeding at any point? Will the radar notice?

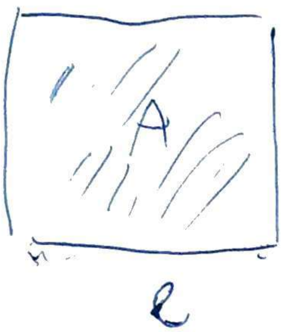


Radar measures the average speed (= average rate of change of position) from t=1 to t=3

$$\frac{f(3) - f(1)}{2} = \frac{45 - 10}{2} = \frac{35}{2} = 17.5 \text{ m/s}$$

- at t_0, the actual speed of the car (= inst. rate of change of position) appears to be big, at t_0 over the speed limit (But: just for a very short period of time) urves.

Exercise: Compute the rate of change of the area of a square depending on the length of its side l, when $l = 3.5$ m.



~~A~~ area $A(l) = l^2$

rate of change of the area $= \frac{dA}{dl} = 2 \cdot l$

• rate of change when $l = 3.5$

$= A'(3.5) = 7 \ [m^2/m]$

Exercise: Consider a pendulum on a train. Its position in the horizontal direction (in m) is described as the function of time (in s) by:

$(t^1)' = 1 \cdot t^0 = 1$

$$s(t) = \frac{1}{2}\sin(t) + 53t$$

(a) Describe its velocity $v(t)$ (in the hor. direction) as a function of time.
(b) What is its average speed over the first 10 seconds?

(a) $v(t) = s'(t) = \frac{1}{2}\cos(t) + 53 \ [m/s]$

(b) average speed = average rate of change of $s(t)$ from $t = 0$ to $t = 10$

$= \frac{s(10) - s(0)}{10} = \frac{\frac{1}{2}\sin(10) + 53 \cdot 10 - \frac{1}{2}\sin(0) - 53}{10}$

$= \frac{1}{20}\sin(10) + 53 - 0 - 0 = \frac{1}{20}\sin(10) + 53$

$\approx 53.009 \ [m/s]$

Exercise: We throw a ball vertically in the air, and as a result, its position function (in m, depending on time in s) is:

$$s(t) = 9t - t^2$$

no velocity for a moment

(a) What is its velocity function?
(b) At what time does the ball reach its highest point, and how high is it?
(c) At what time does the ball hit the ground, and with what speed?

(a) $v(t) = s'(t) = 9 - 2t$

(b) When the ball is at the top, it's velocity is $\underline{v = 0}$

→ to find the time, we solve

$$v(t) = 0$$
$$9 - 2t = 0$$
$$9 = 2t$$
$$\underline{t = 4.5 \text{ s}}$$

How high is the ball? $s(4.5) = 9\cdot(4.5) - (4.5)^2 = \underline{20.25 \text{ m}}$

(c) The ball hits the ground when $\underline{s(t) = 0}$

→ $9t - t^2 = 0$

$(9 - t)\cdot t = 0 \rightarrow \underbrace{t = 0}$ or $t = 9$ ← this is when it fell back down.

this is when the ball was thrown up.

→ $\underline{t = 9}$ s

the speed at the moment is $\underline{v(9) = -9 \text{ m/s}}$

(interpretation: speed is 9 m/s, but in the direction down, ie opposite to the direction of the throw) nondireco

Exercise (time permitting): A company's expected profit P (in thousands of dollars) is estimated to be dependent on the amount a of money spent of advertisment (in thousands of dollars) as follows:

$$P(a) = 200\cancel{\sqrt{a} - a^2} - a \qquad P(a) = 200\sqrt{a} - 0.1a^2 - a$$

(assuming $0 \le a \le 130$).

(a) What is the rate of change of the profit if the company spends $a = 25$ thousands of dollars on advertising?

(b) What is the rate of change of the profit if the company spends $a = 100$ thousands of dollars on advertising?

$$P'(a) = \frac{d}{da}\left(200a^{\frac{1}{2}} - 0.1a^2 - a\right) =$$

$$= 200 \cdot \frac{1}{2} a^{-\frac{1}{2}} - 0.2a - 1 = \frac{100}{\sqrt{a}} - 0.2a - 1$$

(a) $P'(25) = \frac{100}{\sqrt{25}} - (0.2)\cdot 25 - 1 = \frac{100}{5} - 5 - 1 = 20 - 6 = 14$ [thousands of dollars/thousands of dollars]

[interpretation: ~~When we keep~~: we may increase the advertising money (at least a little bit) to increase profit.]

(b) $P'(100) = \frac{100}{\sqrt{100}} - (0.2)\cdot 100 - 1 = 10 - 20 - 1 = -11$

[interpretation: We spent too much on advertising - if we spend slightly less, the profite will increase!]