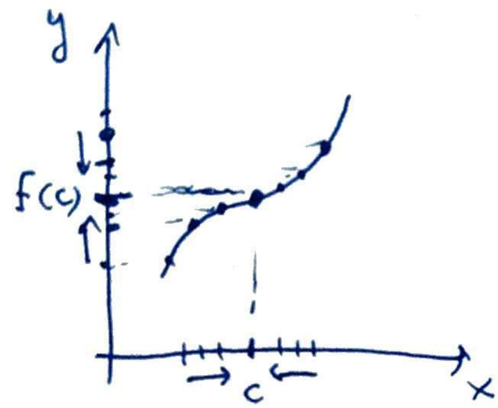
# MA 16010 Lesson 5: (Dis)Continuity

**Continuity**

A function $f(x)$ is continuous at $x = c$ if:

- $f(c)$ is defined,
- $\lim_{x\to c} f(x)$ exists,
- $f(c) = \lim_{x\to c} f(x)$.

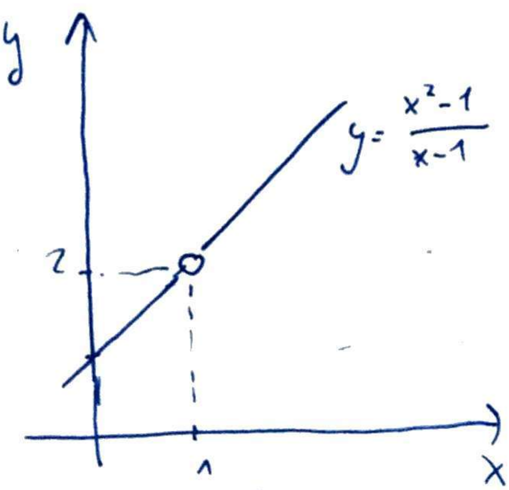


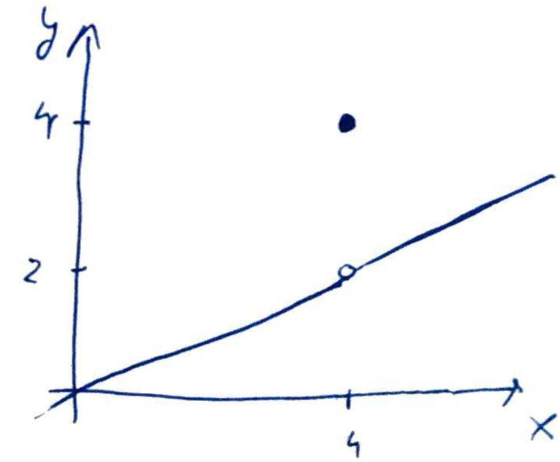

What can happen when the function is **not** continuous?

**1. Holes.** = ~~finite~~ $\lim_{x\to c} f(x)$ exists and is finite.

**Examples:**

(a) $f(x) = \dfrac{\overbrace{x^2-1}^{(x-1)(x+1)}}{x-1}$ $\left(= x+1 \text{ when } x \neq 1\right)$ (b) $f(x) = \begin{cases} \frac{1}{2}x, & x \neq 4 \\ 4, & x = 4 \end{cases}$




discontinuity: hole

~~[illegible]~~ $\lim_{x\to 1} f(x)$ exists, $(=2)$

but it is not equal to ~~[illegible]~~ $f(1)$

($\because$ $f(1)$ is not defined!)

discontinuity: hole

$\lim_{x\to 4} f(x)$ exists $\left(= \frac{1}{2}\cdot 4 = 2\right)$

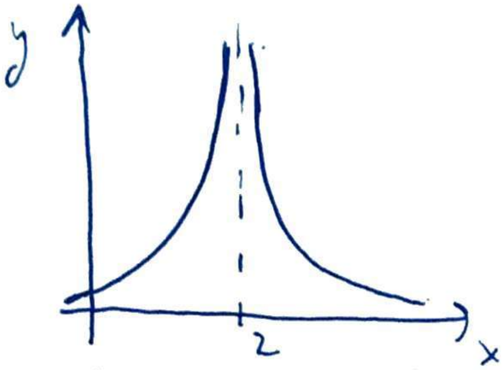
$f(4) = 4$ is defined,

but $\lim_{x\to 4} f(x) \neq f(4)$

**2. Vertical asymptotes.** = at least one of the one-sided limits $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$ exists and is $\pm\infty$

**Examples:**

(a) $f(x) = \dfrac{1}{(x-2)^2}$

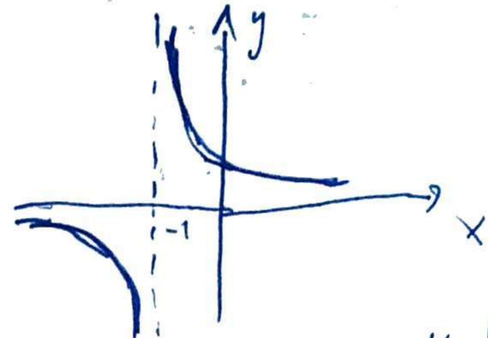



$\lim_{x\to 2^-} \frac{1}{(x-2)^2} = \infty$

$\lim_{x\to 2^+} \frac{1}{(x-2)^2} = \infty$

$\Big\}\ \lim_{x\to 2} \frac{1}{(x-2)^2} = \infty$ so it cannot be equal to $f(2)$ (undefined!)

(b) $f(x) = \dfrac{1}{x+1}$




$\lim_{x\to -1^-} \frac{1}{x+1} = -\infty$

$\lim_{x\to -1^+} \frac{1}{x+1} = \infty$

$\Big\}\ \lim_{x\to c} \frac{1}{x+1}$ does not even exist, but ~~one~~ at least one of the one-sided limits is $\pm\infty$

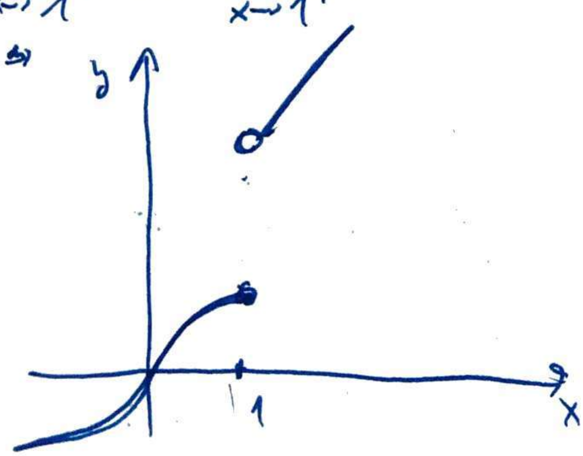
**3. Jumps.** = finite but different one-sided limits.

**Examples:**

(a) $f(x) = \begin{cases} \sqrt[3]{x}, & x \le 1 \\ 2x+1, & x > 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt[3]{x} = \sqrt[3]{1} = 1$

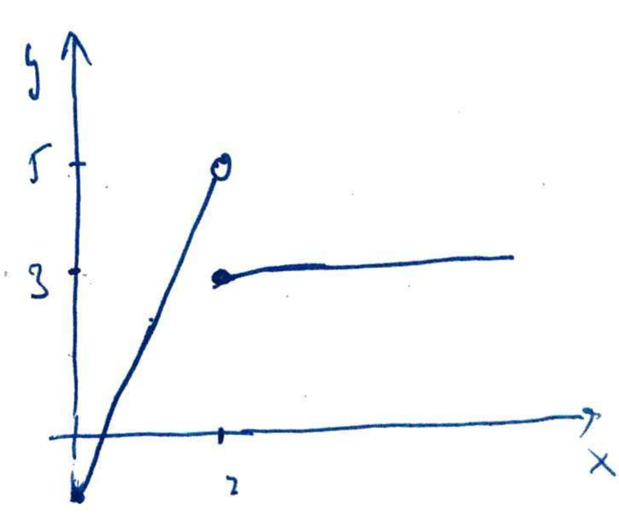
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x+1) = 3$




(b) $f(x) = \begin{cases} 3x-1, & x < 2 \\ 3, & x \ge 2 \end{cases}$

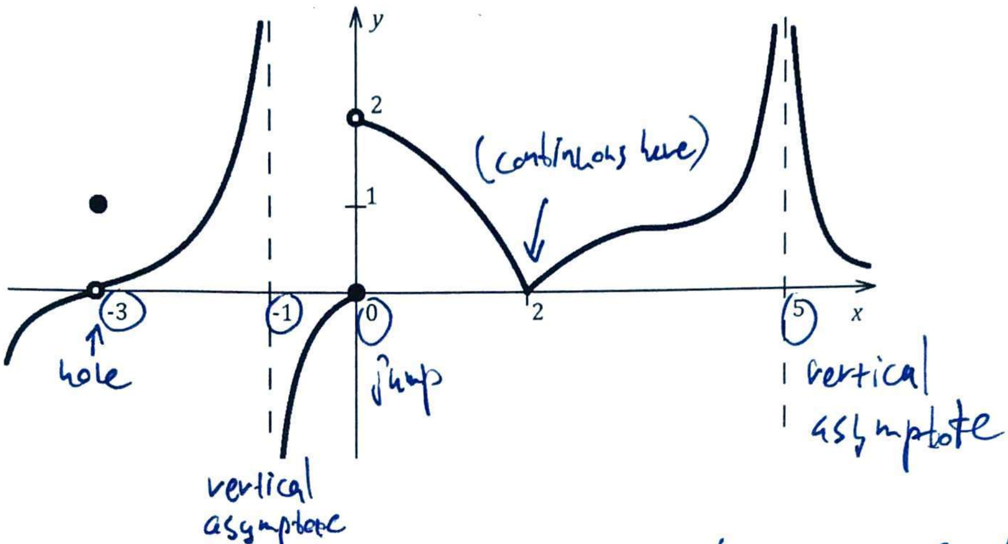
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3x-1) = 5$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3 = 3$



**Exercise:** Find and classify points of discontinuity:

1.




2. $$f(x) = \frac{x^2 + 3x - 10}{x^2 + x - 6}$$

← issue can be only where the function is not defined

ie when $x^2 + x - 6 = 0$

$x_{1,2} = \frac{-1 \pm \sqrt{1 + 4 \cdot 6}}{2}$

$x = 2$, or $x = -3$

$x_{1,2} = \frac{-1 \pm 5}{2} = \begin{cases} -3 \\ 2 \end{cases}$

1) $x = 2$: numerator $= 2^2 + 3 \cdot 2 - 10 = 0$

→ limit "$0/0$"

try factoring numerator: $x^2 + 3x - 10 = (x-2)(x+5)$

$\rightarrow \lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+5)}{\cancel{(x-2)}(x+3)} = \lim_{x \to 2} \frac{x+5}{x+3} = \frac{7}{5}$

hole

2) $x = -3$ $\lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{x+5}{x+3} =$

$= \lim_{x \to -3} (x+5) \cdot \lim_{x \to -3} \frac{1}{x+3} = 2 \cdot \lim_{x \to -3} \frac{1}{x+3}$

→ ~~DNE~~,

DNE, but one-sided limits

are $\lim_{x \to -3^-} \frac{1}{x+3} = -\infty$,

$\lim_{x \to -3^+} \frac{1}{x+3} = \infty$

vertical asymptote

3. $$f(x) = \begin{cases} 2x, & x \leq 2 \\ x^2, & 2 < x \leq 4 \\ \sqrt{x} + 1, & x > 4 \end{cases}$$

$x = 2$ $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2x = 4$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 4$

$\Big\}$ $\lim_{x \to 2} f(x) = 4$

$f(2) = 4$

continuous!

$x = 4$ $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} x^2 = 16$

$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x} + 1 = 3$

$\Big\}$ → jump

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