MA 16010 Lesson 3: Limits Analytically

Preview: Continuity

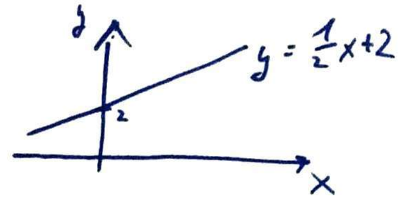
A function $f(x)$ is continuous at $x = c$ if:

- $f(c)$ is defined,
- $\lim_{x \to c} f(x)$ exists,
- $f(c) = \lim_{x \to c} f(x)$.

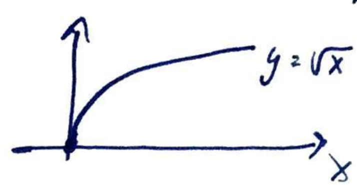
Rule of thumb: Functions defined by a single formula are continuous at every point where they are defined
(if they are also defined around the point).

Examples:

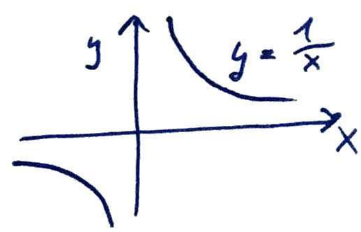
- $f(x) = \frac{1}{2}x + 2$: continuous at all $x \in \mathbb{R}$



- $f(x) = \sqrt{x}$: continuous at all $x > 0$
  at $x = 0$, it is „right continuous"



- $f(x) = \frac{1}{x}$: continuous at all $x > 0$, $x < 0$
  at $x = 0$; nothing meaningful can be said
  ($\frac{1}{x}$ is not defined there)



We can use this implicit continuity to compute simple limits quickly:

Example: Compute:

$$\lim_{x \to -1} \frac{x-1}{x-2} = \frac{(-1)-1}{(-1)-2} = \frac{-2}{-3} = \frac{2}{3} \quad //$$

$f(x) = \dfrac{x-1}{x-2}$ is defined, hence also continuous at $x = -1$ $\rightsquigarrow$ can compute limit by plugging in.

Computational rules for limits.

Assuming that $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist, we have:

- $\lim_{x\to c}(f(x) \pm g(x)) = \left(\lim_{x\to c} f(x)\right) \pm \left(\lim_{x\to c} g(x)\right)$
- $\lim_{x\to c}(f(x) \cdot g(x)) = \left(\lim_{x\to c} f(x)\right) \cdot \left(\lim_{x\to c} g(x)\right)$
- $\lim_{x\to c}(f(x)/g(x)) = \left(\lim_{x\to c} f(x)\right) / \left(\lim_{x\to c} g(x)\right)$
- $\lim_{x\to c}(k \cdot f(x)) = k \cdot \lim_{x\to c} f(x)$
- $\lim_{x\to c}(f(x)^n) = \left(\lim_{x\to c} f(x)\right)^n$

as long as the expressions on the right–hand side make sense.

(**Example:**

makes sense: $1+1$, $\frac{0}{5}$, $2+\infty$, $2-\infty$, $\frac{0}{\infty}$ $(=0)$, $\infty\cdot(-\infty)$ $(=-\infty)$...

does not make sense: $\infty-\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{\infty}{-\infty}$, ...

)

Example: Using that $\lim_{x\to 0} \frac{\sin(x)}{x} = \underline{\ 1\ }$, compute:

$$\lim_{x\to 0}\left(\frac{x+2}{x}\cdot\sin(x)+4x\right) = \lim_{x\to 0}\left(\frac{x+2}{x}\cdot\sin x\right) + \underbrace{\lim_{x\to 0} 4x}_{=4\cdot 0=0}$$

(plugging in, by continuity)

$$= \underbrace{\left(\lim_{x\to 0}(x+2)\right)}_{=2} \cdot \underbrace{\left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)}_{=1} = 2\cdot 1 = \underline{\underline{2}}$$

(plugging in, continuity)

More complicated limits:

1. Type "0/0":

Strategy: Try to factor out „(x-c)" both from numerator & denominator, and cancel out. Then compute the limit

Example:

$$\lim_{x\to 2}\frac{x^2-x-2}{x^2-2x} = \lim_{x\to 2}\frac{x+1}{x} \overset{\text{plug in}}{=} \frac{3}{2}$$

$$\left[\begin{array}{l} x^2-x-2=(x-2)(x+1) \\ x^2-2x=(x-2)\cdot x \\ \rightsquigarrow \dfrac{x^2-x-2}{x^2-2x}=\dfrac{(x-2)(x+1)}{(x-2)x}=\dfrac{x+1}{x} \text{ for } x\neq 2 \end{array}\right]$$

2. Type "(finite number)/0":

Strategy: investigate $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$ separately. Typically they will yield $+\infty$ or $-\infty$. Then conclude about the overall limit ($=\infty$, $=-\infty$ or DNE)

Example:

$$\lim_{x\to 3}\frac{3x+2}{x-3} = \lim_{x\to 3}(3x+2)\cdot\lim_{x\to 3}\frac{1}{x-3} = 11\cdot\lim_{x\to 3}\frac{1}{x-3}$$

1) $\lim_{x\to 3^-}\frac{1}{x-3}$: when $x<3$ with x ~~-3 small in magnitude~~ x close to 3,

$x-3$ is negative, small in magnitude

$\to \frac{1}{x-3}$ is negative, big in magnitude

$\to \lim_{x\to 3^-}\frac{1}{x-3} = -\infty$

2) $\lim_{x\to 3^+}\frac{1}{x-3}$: when $x>3$, x close to 3, $x-3$ is positive small

$\to \frac{1}{x-3}$ is positive big. So $\lim_{x\to 3^+}\frac{1}{x-3}=\infty$

overall limit DNE $\Rightarrow \lim_{x\to 3}\frac{3x+2}{x-3}$ DNE.

Example:

$$\lim_{x\to 3}\frac{3x+2}{(x-3)^2} = \lim_{x\to 3}(3x+2)\cdot\lim_{x\to 3}\frac{1}{(x-3)^2} = 11\cdot\lim_{x\to 3}\frac{1}{(x-3)^2}$$

• $\lim_{x\to 3^-}\frac{1}{(x-3)^2}$: $x<3$, close $\to$ $x-3$ negative small $\to$ $\frac{1}{(x-3)^2}$ positive, big

$\to \lim_{x\to 3^-}\frac{1}{(x-3)^2}=\infty$

• $\lim_{x\to 3^+}\frac{1}{(x-3)^2}$: $x>3$, close to 3 $\to$ $x-3$ positive & small $\to$ $\frac{1}{(x-3)^2}$ positive, big

$\to \lim_{x\to 3^+}\frac{1}{(x-3)^2}=\infty$ $\to$ $\lim_{x\to 3}\frac{1}{(x-3)^2}=\infty$, so $\lim_{x\to 3}\frac{3x+2}{(x-3)^2}=\infty$

3. Function defined piecewise by formulas:

Example: Consider $f(x)$ defined by

$$f(x)=\begin{cases}3x+3, & x\le 0\\ 1+\frac{4}{x}, & 0<x\le 2\\ \frac{3x^2-6x}{2x-4}, & x>2.\end{cases}$$

Strategy: in "borderline cases" (e.g. when $x\to 0$ or $x\to 2$ for the above function), investigate $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$ separately by previous methods

Example (continued): For the function $f(x)$ defined above, find:

(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$ (d) $f(0)$

(e) $\lim_{x\to 2^-} f(x)$ (f) $\lim_{x\to 2^+} f(x)$ (g) $\lim_{x\to 2} f(x)$ (h) $f(2)$

(a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(3x+3) = 3\cdot 0+3 = 3$

(b) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(1+\frac{4}{x}\right) = \infty$

($x\to 0$ from right $\Rightarrow$ x is small positive $\Rightarrow$ $\frac{1}{x}$ is big and positive, goes to ∞)

(c) $\lim_{x\to 0} f(x)$ DNE $\left(\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)\right)$

(d) $f(0) = 3\cdot 0+3 = 3$

(e) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(1+\frac{4}{x}\right) = 1+\frac{4}{2} = 3$

(f) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\frac{3x^2-6x}{2x-4}$

$= \lim_{x\to 2^+}\frac{3x(x-2)}{2(x-2)} = \lim_{x\to 2^+}\frac{3x}{2} = \frac{3\cdot 2}{2} = 3$

(g) $\lim_{x\to 2} f(x) = 3$ $\left(\lim_{x\to 2^-} = \lim_{x\to 2^+}\right)$

(h) $f(2) = 1+\frac{4}{2} = 3$