MA 16010 Lesson 35: Exponential Growth

Recall: A differential equation is an equation involving $t, y = y(t), y', y'' \dots$

We studied $y' = f(t)$ (e.g. $y' = 3t+1$)

$y'' = g(t)$ (e.g. $y'' = \sin(t) + 2, \dots$)

Today we consider equations of the form

$$\boxed{y' = ky} \quad (k \text{ is a constant}). \qquad (*)$$

Observe/recall:

- $y = e^t$ $\quad \frac{d}{dt}[e^t] = e^t \Rightarrow y = e^t$ is a solution of the eqn $y' = y$
- $y = e^{kt}$ $\quad \frac{d}{dt}[e^{kt}] = e^{kt} \cdot k$ (chain rule) $\Rightarrow y = e^{kt}$ is a solution of $y' = ky$
- $y = Ce^{kt}$, C a constant $\dots$ $\frac{d}{dt}[Ce^{kt}] = C \cdot e^{kt} \cdot t$

→ also $y = C \cdot e^{kt}$ for any C is a solution to $y' = ky$.

The general solution to the equation $(*)$ is: $y = C \cdot e^{kt}$, C a constant

Exercise: Solve the initial value problem $\dfrac{dy}{dt} = 3y, \quad y(0) = 15.$

general solution: $y = C \cdot e^{3t}$

→ to solve the i.v.p., we use $y(0) = 15$:

$$C \cdot e^{3 \cdot 0} = 15$$

$$C \cdot 1 = 15$$

$$C = 15 \rightarrow y = 15 \cdot e^{3t}$$

Exercise: Given that $\frac{dy}{dt} = 6y, \quad y(6) = 20,$ find $y(10)$.

general solution: $y = C \cdot e^{6t}$

particular solution: $y(6) = 20$

$C \cdot e^{6 \cdot 6} = 20$

$C \cdot e^{36} = 20$

$C = 20 \cdot e^{-36}$

$y(t) = 20 \cdot e^{-36} \cdot e^{6t} =$

$= 20 \cdot e^{6t-36}$

$y(10) = 20 \cdot e^{6 \cdot 10 - 36} = 20 e^{24}$

Exponential growth model: = situation modelled by the equation

$y' = ky$, where k is a positive constant

Solutions are of the form $y = C \cdot e^{kt}$, usually with $C > 0$

Exercise: The population of a culture of bacteria, $P(t)$, where t is time in days, is growing at a rate proportional to the population. The growth rate is 0.3 . If the initial population is $P(0) = 1000$,

(a) how big is the population after 10 days?

$\frac{dP}{dt} = 0.3P$, $P(0) = 1000$

$\rightarrow P(t) = C \cdot e^{0.3t}$

$P(0) = 1000$

$C \cdot e^{(0.3)0} = 1000$

$C = 1000$

$P(t) = 1000\, e^{0.3t}$

$P(10) = 1000\, e^{3}$

$\approx 20\,086$

(b) how long will it take for the population to double?

$P(t) = 1000\, e^{0.3t}$

want: t such that $P(t) = 2000$

$1000\, e^{0.3t} = 2000$

$e^{0.3t} = \frac{2000}{1000} = 2 \quad / \ln(\ldots)$

$0.3t = \ln(2)$

$t = \frac{\ln(2)}{0.3} \approx 2.3$ days.

Exercise: John currently has \$ 8000 on a savings account at Bank A. On his account, the interest is compounded continuously, with the annual rate of interest 4.5%.

(a) How much will be in the account after 9 years? Round to nearest cent.

A = amount of dollars

$$\frac{dA}{dt} = 0.045A \quad A(0) = 8000$$

$$A = C \cdot e^{0.045t}$$

$$A(0) = 8000$$

$$C \cdot e^{0} = 8000$$

$$C = 8000$$

$$A(t) = 8000\, e^{0.045t}$$

$$A(9) = 8000 \cdot e^{(0.045) \cdot 9}$$

$$\approx \$\ 11\,994.42$$

(b) John also has \$ 10000 on an account at Bank B, also compounded continuously. The bank guarantees that this amount will grow to \$12500 after 7 years. What is the annual interest rate?

~~A~~ B = amount of dollars

$\frac{dB}{dt} = r \cdot B$ ... want to find the rate r

$$B(t) = C \cdot e^{rt}$$

$$B(0) = 10\,000$$

$$C \cdot e^{r \cdot 0} = 10000$$

$$C = 10\,000$$

$$B(t) = 10000\, e^{rt}$$

$$B(7) = 12500$$

$$10\,000\, e^{r \cdot 7} = 12\,500$$

$$\leadsto r = \frac{\ln(1.25)}{7} \approx 0.032,$$

so 3.2%