MA 16010 Lesson 34: Numerical integration

Sometimes it is not practical/possible to evaluate integrals "analytically".

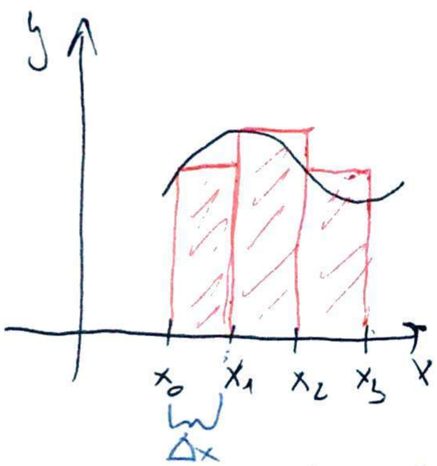
Examples:

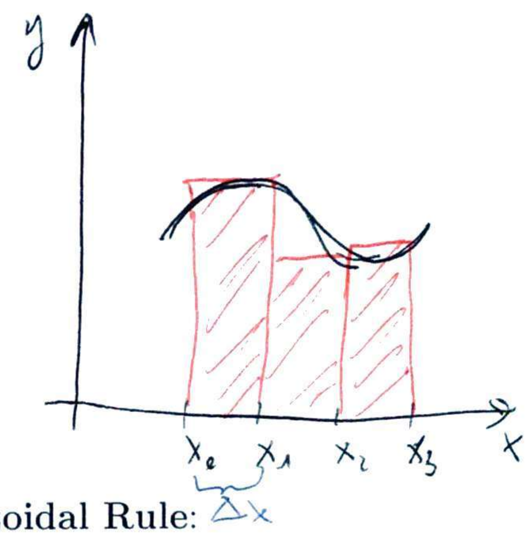
$$\int \frac{\ln(1+t)}{(1+t)^2}\,\mathrm{d}t$$ ... possible to compute, more involved (MA 16020)

$$\int e^{x^2}\,\mathrm{d}x$$ ... no formula for this antiderivative exists!

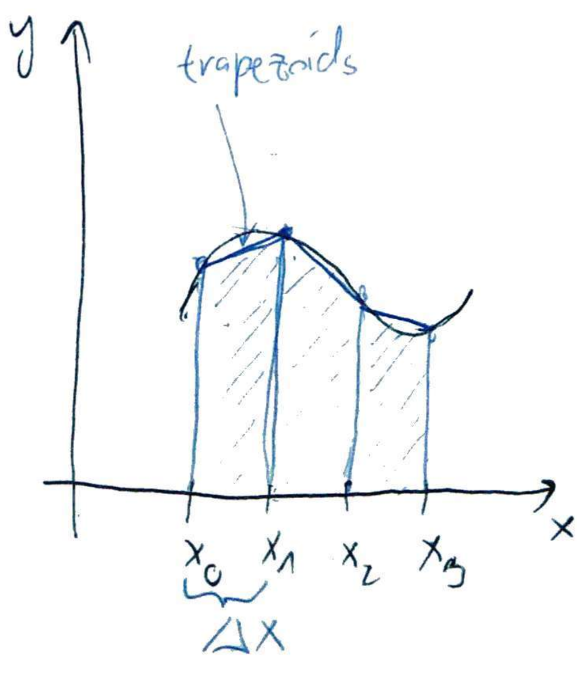
⇝ in practice, one often uses **numerical methods/approximations** to evaluate definite integrals.

Numerical method that we have seen already is the method of left/right Riemann sums:

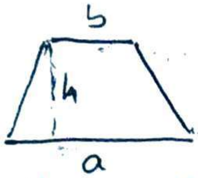




An improvement upon this is the **Trapezoidal Rule**:

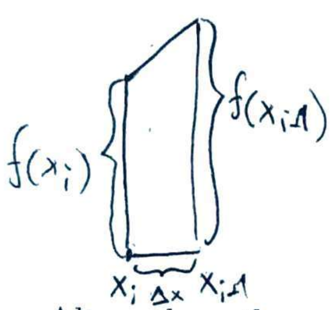


Recall: How to compute the area of a trapezoid:



$$\text{Area} = \frac{1}{2} h \cdot (a+b)$$

In the case of trapezoids from the previous picture, we get:



$$\begin{pmatrix} f(x_{i+1}) = a \\ f(x_i) = b \\ \Delta x = h \end{pmatrix} \rightsquigarrow \text{area} = \frac{1}{2} \Delta x \cdot \left(f(x_i) + f(x_{i+1})\right)$$

Altogether, the approximation of the integral is:

$$T_n = \frac{1}{2}\Delta x \left(f(x_0) + f(x_1)\right) + \frac{1}{2}\Delta x \left(f(x_1) + f(x_2)\right) + \dots + \frac{1}{2}\Delta x \left(f(x_{n-1}) + f(x_n)\right)$$

$$\boxed{T_n = \frac{1}{2}\Delta x \left(f(x_0) + 2 \cdot f(x_1) + 2 \cdot f(x_2) + \dots + 2 \cdot f(x_{n-1}) + f(x_n)\right)}$$

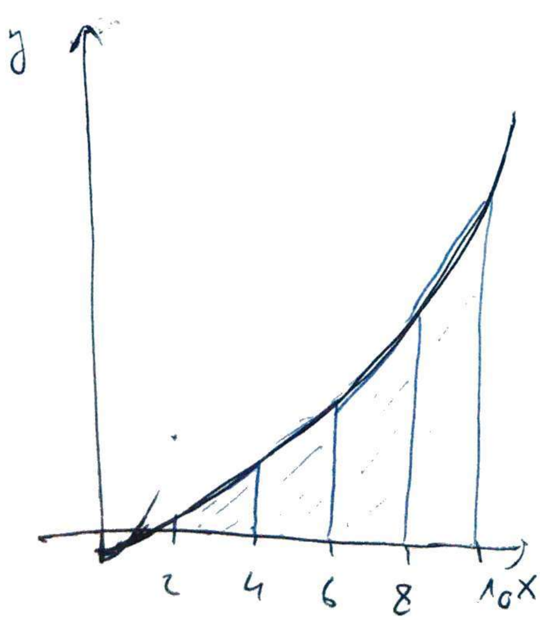
Example:

(A) Using the Trapezoidal Rule with 4 trapezoids, approximate the integral

$$\int_2^{10} (x^2 - 1)\, dx.$$

$$\Delta x = \frac{10-2}{4} = 2$$

x_i	2	4	6	8	10
$f(x_i)$	3	15	35	63	99



$$T_4 = \frac{1}{2} \cdot 2 \cdot \left(3 + 2 \cdot 15 + 2 \cdot 35 + 2 \cdot 63 + 99\right)$$

$$= 3 + 30 + 70 + 126 + 99$$

$$= 229 + 99 = \underline{\underline{328}}$$

(B) Compute the integral precisely and compare:

$$\int_2^{10} (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_2^{10} = \left(\frac{1000}{3} - 10\right) - \left(\frac{8}{3} - 2\right)$$

$$= \underline{\underline{\frac{968}{3}}} \approx 322.67$$

compare:

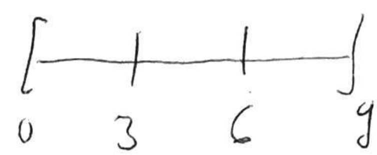
True area: ≈ 322.67

trapezoidal rule: 328 ← much better!

(from Quiz 11) $\begin{cases} \text{Left RS: } 232 \\ \text{Right RS: } 424 \end{cases}$

Exercise: Using the Trapezoidal Rule with $n = 3$, approximate the integral $\int_0^9 (e^{x^2} - 1)\,dx$.

$$\Delta x = \frac{9-0}{3} = 3$$



x_i	0	3	6	9
$f(x_i)$	0	$e^9 - 1$	$e^{36} - 1$	$e^{81} - 1$

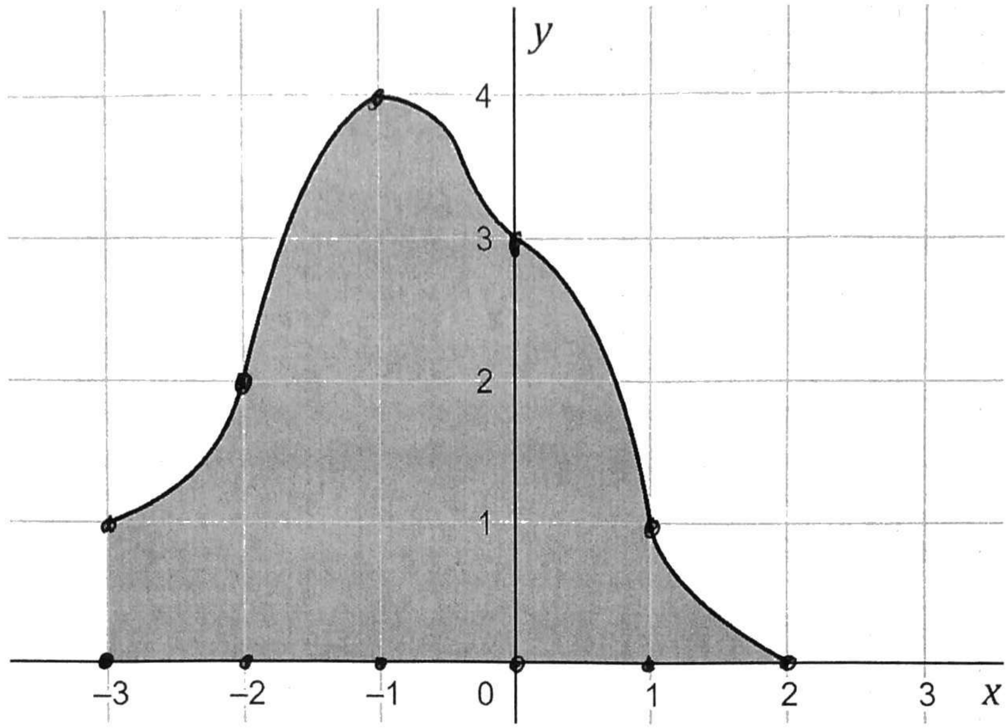
$$T_3 = \frac{1}{2} \cdot 3 \cdot \left(0 + 2(e^9 - 1) + 2(e^{36} - 1) + 1 \cdot (e^{81} - 1)\right)$$

$$= \frac{3}{2}\left(2e^9 + 2e^{36} + e^{81} - 2 - 2 - 1\right)$$

$$= \underline{\underline{3e^9 + 3e^{36} + \frac{3}{2}e^{81} - \frac{15}{2}}} \approx \text{a horribly big number}$$

$$\left(2.259145971 \cdot 10^{35}\right)$$

Exercise: Using the Trapezoidal Rule with $n = 5$, approximate the area of the shaded region below.



$$\Delta x = \frac{2-(-3)}{5} = \frac{5}{5} = 1$$

x_i	-3	-2	-1	0	1	2
$f(x_i)$	1	2	4	3	1	0

$$T_5 = \frac{1}{2} \cdot 1 \cdot \left(1 + 2\cdot 2 + 2\cdot 4 + 2\cdot 3 + 2\cdot 1 + 0\right)$$

$$= \frac{1}{2}\left(1 + 4 + 8 + 6 + 2\right) = \frac{21}{2}$$