MA 16010 Lesson 33: Fundamental Theorem of Calculus II

Recall (fundamental theorem of calculus): If $y = f(x)$ is a function that is continuous on $[a, b]$ then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b \left(= F(x)\Big|_a^b\right) = F(b) - F(a),$$

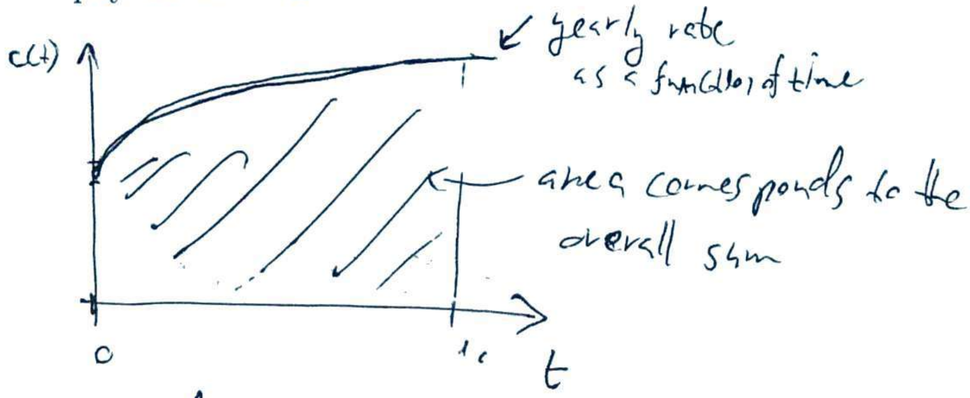
where $F(x)$ is the antiderivative of $f(x)$, ie $\int f(x)\,dx = F(x) + C$

What are definite integrals good for:
They measure how certain qunatities "accumulate over time".

Example: A company maintains a website through a web hosting provider. The expected hosting cost per year, in dollars per year, is

$$c(t) = 500 + 300\sqrt{t}$$

where t is time in years from now. How much does the company expect to pay for hosting of their website over the next 10 years?



$$\text{Cost} = \int_0^{10} \left(500 + 300\sqrt{t}\right) dt = \int_0^{10} \left(500 + 300t^{\frac{1}{2}}\right) dt = \left[500t + 200t^{\frac{3}{2}}\right]_0^{10}$$

$$= \left(500 \cdot 10 + 200 \cdot 10^{3/2}\right) - 0 =$$

$$= 5000 + 200 \cdot 10^{3/2} \approx 11\,324.6\ \$$$

Exercise: A water pipe bursts in the bathroom at 9:05 am, and as a result, water floods the bathroom at the rate

$$r(t) = 70 + 3\sqrt{t} \text{ gal/min},$$

where t is the time in minutes since 9:05 am.

(a) How much water poured into the bathroom within the first hour, i.e. between 9:05 am and 10:05 am?

(b) How much water poured into the bathroom during the second hour, i.e. between 10:05 am and 11:05 am?

9:05 am ... $t=0$
10:05 am ... $t=60$

(a) Amount of water $= \int_0^{60} (70+3\sqrt{t})\,dt$

$= \left[70t + 2t^{3/2}\right]_0^{60} = \left(70\cdot 60 + 2\cdot(60)^{3/2}\right) - 0 =$

$= 4200 + 2\cdot(60)^{3/2} \approx 5129.5$ gal

(b) ~~t=~~ 10:05 am ... $t=60$
11:05 am ... $t=120$

$\rightsquigarrow$ Amount of water $= \int_{60}^{120} (70+3\sqrt{t})\,dt = \left[70t + 2t^{3/2}\right]_{60}^{120}$

$= \left(70\cdot 120 + 2\cdot(120)^{3/2}\right) - \left(70\cdot 60 + 2\cdot(60)^{3/2}\right)$

$= 4200 + 2\left(120^{3/2} - 60^{3/2}\right) \approx 5899.6$ gal

Exercise: The acceleration of a car t seconds after hitting the brakes is

$$a(t) = -5 - \frac{t}{5}$$

miles per hour per second.

(a) What is the decrease in velocity 5 seconds after after hitting the brakes?

(b) Given that the initial velocity of the car was 50 mph, how much time is needed before the car comes to a halt?

(a) decrease in velocity over the first 5 seconds

$$\Delta v = \int_0^5 a(t)\,dt = \int_0^5 \left(-5 - \frac{t}{5}\right) dt = \left[-5t - \frac{t^2}{10}\right]_0^5 =$$

$$= \left(-5\cdot 5 - \frac{25}{10}\right) - 0 = -25 - \frac{25}{10} = \frac{-275}{10} = -27.5 \text{ mph}$$

(b) decrease in velocity over the first x seconds

$$\Delta v = \int_0^x a(t)\,dt = \int_0^x \left(-5 - \frac{t}{5}\right) dt = \left[-5t - \frac{t^2}{10}\right]_0^x =$$

$$= \left(-5x - \frac{x^2}{10}\right) - 0 = -5x - \frac{x^2}{10} \text{ mph}$$

need $\Delta v = -50$ mph, so that overall velocity $= 0$.

$\rightsquigarrow$ $-5x - \frac{x^2}{10} = -50$ $\qquad x_{1,2} = \frac{-50 \pm \sqrt{2500 + 2000}}{2}$

$\frac{x^2}{10} + 5x - 50 = 0$ $\qquad x_{1,2} = \frac{-50 \pm \sqrt{4500}}{2} = \frac{-50 \pm 30\sqrt{5}}{2} =$

$x^2 + 50x - 500 = 0$

$$= -25 \pm 15\sqrt{5}$$

$x_1 = -25 + 15\sqrt{5} > 0$

$x_2 = -25 - 15\sqrt{5} < 0$

$\rightarrow$ time is $-25 + 15\sqrt{5}$ s ≈ 8.5 s