MA 16010 Lesson 28: Antiderivatives II

A **differential equation** (in x and y) is an equation relating $x, y = y(x)$, and the derivatives $y', y'', \ldots$

Examples:

1) $y' = 3x + 5$,
2) $2\cos(x) + y'' = 1$,
3) $3x^2y' - 2xy = x^{3/2}$ (more complicated – Calc 2), ...

Exercise: Find the general solution to the equations 1) and 2) above.

1) $y' = 3x + 5$

$y = \int (3x+5)\,dx = 3\frac{x^2}{2} + 5x + C$ C a constant

2) $2\cos(x) + y'' = 1$

$y'' = 1 - 2\cos(x)$

$y' = \int (1 - 2\cos x)\,dx = x - 2\sin x + C$

$y = \int (x - 2\sin x + C)\,dx = \frac{x^2}{2} + 2\cos(x) + Cx + D$

C, D are arbitrary constants

[general solutions]

To pinpoint one particular solution, one can specify an additional value of y (and y' for example) at a point. This is called an **initial value problem.** ("$y(3) = 1$")

To solve them:
- solve the diff. equation by integrating (depends on some constant C)
- plug in your solution to the initial condition ⇝ equation for C ⇝ solve for C.
- repeat if necessary.

Exercise: Solve the initial value problem

$$y' = 5 - 4x, \quad y(2) = 5$$

1) general solution: $y' = 5 - 4x$

$y = \int (5-4x)\,dx = 5x - 2x^2 + C$

2) determine the value of C: $y(2) = 5$

$5 \cdot 2 - 2 \cdot 2^2 + C = 5$

$2 + C = 5$

$C = 5 - 2 = 3$

$y = 5x - 2x^2 + 3$

Exercise: Given that $y = y(x)$ satisfies

$$y'' = 3e^x - 2, \quad y'(0) = 4, \quad y(0) = 8,$$

find $y(2)$. ← to obtain this value, we will find the whole function y(x) first

1) $y' = \int (3e^x - 2)\,dx = 3e^x - 2x + C$

2) determine C: $y'(0) = 4$

$3 \cdot \underbrace{e^0}_{1} - 2 \cdot 0 + C = 4$

$3 + C = 4 \rightarrow C = 1$

$y' = 3e^x - 2x + 1$

3) $y = \int (3e^x - 2x + 1)\,dx = 3e^x - x^2 + x + D$

4) determine D: $y(0) = 8$

$3 \cdot e^0 - 0^2 + 0 + D = 8$

$3 + D = 8 \rightarrow D = 5$

$y(x) = 3e^x - x^2 + x + 5$

$y(2) = 3 \cdot e^2 - 2^2 + 2 + 5 = 3e^2 + 3$

Exercise: The rate of change dP/dt of a population of rabbits is proportional to the square root of t with proportionality constant 4, where P is the population size and t is the time that passed from the present moment (in months). If the initial size of the population is 500, find the (approximate) population after 5 months.

$$\underbrace{\frac{dP}{dt} = 4\cdot\sqrt{t} \qquad P(0)=500}_{\text{initial value problem}}$$

Want: $P(5)$

$$P = \int 4\sqrt{t}\,dt = \int 4\cdot t^{\frac{1}{2}}\,dt = 4\cdot\frac{t^{3/2}}{3/2} + C$$

$$= \frac{8}{3}\cdot t^{3/2} + C \qquad = 4\cdot\int t^{1/2}\,dt$$

Determine C: $P(0)=500$

$$\underbrace{\frac{8}{3}\cdot 0^{3/2}}_{0} + C = 500$$

$$C = 500$$

$$\rightarrow P(t) = \frac{8}{3}t^{3/2} + 500$$

$$P(5) = \frac{8}{3}\cdot 5^{3/2} + 500 \approx 530 \text{ rabbits}$$