MA 16010 Lesson 27: Antiderivatives I

Example: If $f(x) = 3x^2 + x$ is the derivative of a function $F(x)$, then
$F(x) = x^3 + \frac{1}{2}x^2$

or $x^3 + \frac{1}{2}x^2 + 1$, or $x^3 + \frac{1}{2}x^2 - 1000$, ...

In general, $F(x) = x^3 + \frac{1}{2}x^2 + C$, C general constant

Antiderivatives. An antiderivative of a function $f(x)$ is a function $F(x)$ such that: $F'(x) = f(x)$.

An antiderivative is determined only up to an additive constant. When $F(x)$ is an antiderivative of $f(x)$, we write:
$$\int f(x)\,dx = F(x) + \underbrace{C}_{\text{"integration constant"}}$$

We also call $F(x)$ <u>the indefinite integral of $f(x)$</u> and the process of finding it <u>indefinite integration.</u>.

Rules for integration. We can reverse engineer rules for antiderivatives from those for derivatives:

- **Additivity, constant multiples:**

$$\frac{d}{dx}\left[F(x) \pm G(x)\right] = F'(x) \pm G'(x) \rightsquigarrow \boxed{\int \left(f(x) \pm g(x)\right) dx = \int f(x)\,dx \pm \int g(x)\,dx}$$

$$\frac{d}{dx}\left[k \cdot F(x)\right] = k \cdot F'(x) \rightsquigarrow \boxed{\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx}$$

k is a constant

- (**Product and chain rule:** MA26200, "int. by parts," "substitution")

- **Constant rule:**

$$\frac{d}{dx}[C] = 0 \rightsquigarrow \int 0\,dx = \underbrace{C}_{\text{int. constant}}$$

C a constant

- **Power rule:**

$$\frac{d}{dx}\left[x^n\right] = n \cdot x^{n-1} \rightsquigarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\left(\frac{d}{dx}\left[x^{n+1}\right] = (n+1) \cdot x^n\right)$$

- **Antiderivatives of other functions:**

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\sin(x)\Big] = \cos(x) \quad \rightsquigarrow \quad \int \cos(x)\,\mathrm{d}x = \sin(x) + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\cos(x)\Big] = -\sin(x) \quad \rightsquigarrow \quad \int \sin(x)\,\mathrm{d}x = -\cos(x) + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\ln(x)\Big] = \frac{1}{x},\ x > 0 \quad \rightsquigarrow \quad \int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

when $x<0$ ($|x|=-x$)
$\ln|x| = \ln(-x)$
$\frac{d}{dx}\left[\ln(-x)\right] = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[e^x\Big] = e^x \quad \rightsquigarrow \quad \int e^x\,\mathrm{d}x = e^x + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\tan(x)\Big] = \sec^2(x) \quad \rightsquigarrow \quad \int \sec^2(x)\,\mathrm{d}x = \tan(x) + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\cot(x)\Big] = -\csc^2(x) \quad \rightsquigarrow \quad \int \csc^2(x)\,\mathrm{d}x = -\cot(x) + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\sec(x)\Big] = \sec(x)\tan(x) \quad \rightsquigarrow \quad \int \sec(x)\tan(x)\,\mathrm{d}x = \sec(x) + C$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\csc(x)\Big] = -\csc(x)\cot(x) \quad \rightsquigarrow \quad \int \csc(x)\cot(x)\,\mathrm{d}x = -\csc(x) + C$$

Exercise: Compute

(a) $$\int \frac{4x^3 + \sqrt[5]{x^3}}{x}\,\mathrm{d}x = \int \left(4x^3 + \sqrt[5]{x^3}\right) x^{-1}\,dx = \int \left(4x^3 + x^{3/5}\right) x^{-1}\,dx$$

$$= \int \left(4x^2 + x^{-2/5}\right) dx = 4 \cdot \underbrace{\int x^2\,dx}_{\frac{x^3}{3}(+C)} + \underbrace{\int x^{-2/5}\,dx}_{\frac{x^{3/5}}{3/5}(+C)} =$$

$$= \frac{4}{3}x^3 + \frac{5}{3}x^{3/5} + C$$

Exercise (cont.): Compute

(b) $\int \sec(x)\Big(3\cos(x) - 5\tan(x)\Big)\, \mathrm{d}x = \int \Big(3 \underbrace{\cos(x)\sec(x)}_{=1} - 5\tan(x)\sec(x)\Big) dx$

$$= 3\cdot \int 1\cdot dx - 5\int \tan(x)\sec(x)\, dx =$$

$$= 3\cdot \underbrace{\int x^0\, dx}_{x\,(+C)} - 5\underbrace{\int \tan(x)\sec(x)\, dx}_{\sec(x)\,(+C)} = \underline{\underline{3x - 5\sec(x) + C}}$$

(c) $\int (x-2)^2\, \mathrm{d}x = \int (x^2 - 4x + 4)\, dx = \int x^2 dx - 4\int x\, dx + 4\cdot\int 1\, dx$

$$= \frac{x^3}{3} - 4\cdot\frac{x^2}{2} + 4\cdot x + C$$

$$= \underline{\underline{\frac{x^3}{3} - 2x^2 + 4x + C}}$$

(d) $\int \frac{2 - 3xe^x + \pi x\sin(x)}{x}\, \mathrm{d}x = 2\int \frac{1}{x}\, dx - 3\cdot\int \frac{xe^x}{x}\, dx + \pi\cdot\int \frac{x\sin x}{x}\, dx =$

$$= 2\cdot\underbrace{\int \frac{1}{x}\, dx}_{\ln|x|\,(+C)} - 3\cdot\underbrace{\int e^x\, dx}_{e^x\,(+C)} + \pi\cdot\underbrace{\int \sin x\, dx}_{-\cos(x)\,(+C)}$$

$$= \underline{\underline{2\ln|x| - 3e^x - \pi\cos(x) + C}}$$