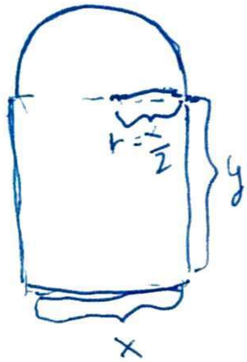
MA 16010 Lesson 25: Optimization II

Recall (Optimization step–by–step):

1. draw a picture, set up variables
2. find the quantity to optimize
3. find constraints
4. $\boxed{2.}$ & $\boxed{3.}$ $\rightsquigarrow$ objective function to maximize/minimize
5. critical points of the obj. function (derivative $=0$)
6. compute all the other quantities the problem asks for

Exercise: A norman window has the shape of a semicircle on top of a rectangle. If the perimeter of the Norman window is to be 4 m, find the dimensions (of the rectangle) that admits the most light.



Want: maximize area $A = x \cdot y + \frac{1}{2}\pi r^2$

$= x \cdot y + \frac{1}{2}\pi \cdot \left(\frac{x}{2}\right)^2 \qquad \left(\left(\frac{x}{2}\right)^2 = \frac{x^2}{4}\right)$

$= x \cdot y + \frac{1}{8} \cdot \pi \cdot x^2$

Constraints: perimeter $= 4$m: $\quad x + 2y + \pi \frac{x}{2}$ $\quad$ (recall: perimeter circle $= 2 \cdot \pi \cdot r$)

$x + 2y + \frac{\pi}{2}x = 4$

$2y = 4 - x - \frac{\pi}{2}x$

$y = \frac{1}{2}\left(4 - x - \frac{\pi}{2}x\right) = 2 - \frac{x}{2}\left(1 + \frac{\pi}{2}\right)$

objective function:

$A = x \cdot \frac{1}{2}\left(4 - x - \frac{\pi}{2}x\right) + \frac{1}{8}\pi x^2$

$= 2x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$

$= 2x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$

$A' = 2 - x - \frac{\pi}{4}x$

$2 - x - \frac{\pi}{4}x = 0$

$2 - x\left(1 + \frac{\pi}{4}\right) = 0$

$x\left(1 + \frac{\pi}{4}\right) = 2$

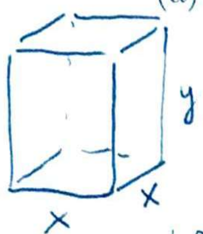
$x = \frac{2}{1 + \pi/4}$ $(\approx 1.12\text{m})$

$y = 2 - \frac{\left(\frac{2}{1 + \frac{\pi}{4}}\right) \cdot \left(1 + \frac{\pi}{2}\right)}{2} =$

$= 2 - \frac{1 + \frac{\pi}{2}}{1 + \frac{\pi}{4}}$ $(\approx 0.56\text{ m})$

s **Exercise (effectivity of container shapes):** We have $3\,\mathrm{m}^2$ of material to construct a container. What maximal volume of the container can be achieved, assuming its shape is:

(a) box with square base:



Want: maximize volume $V = x^2 \cdot y$

Constraint: surface area = 3 $\quad 2x^2 + 4xy = 3$

$4xy = 3 - 2x^2$

$y = \dfrac{3-2x^2}{4x}$

objective function $V = x^2 \cdot \dfrac{3-2x^2}{4x} = \dfrac{x}{4}(3-2x^2) = \dfrac{3}{4}x - \dfrac{1}{2}x^3$

$V' = \dfrac{3}{4} - \dfrac{3}{2}x^2$

$\dfrac{3}{4} - \dfrac{3}{2}x^2 = 0$

$\dfrac{3}{2}x^2 = \dfrac{3}{4}$ $\qquad x = \pm\sqrt{\dfrac{1}{2}}$

$x^2 = \dfrac{1}{2}$ $\qquad x = \sqrt{\dfrac{1}{2}}$

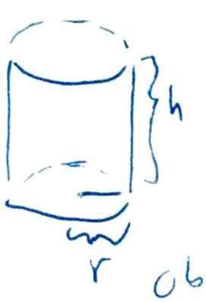
Maximal volume

$V = \dfrac{3}{4}\cdot\sqrt{\dfrac{1}{2}} - \dfrac{1}{2}\cdot\left(\sqrt{\dfrac{1}{2}}\right)^3 = \dfrac{3}{4}\cdot\sqrt{\dfrac{1}{2}} - \dfrac{1}{2}\cdot\dfrac{1}{2}\sqrt{\dfrac{1}{2}}$

$= \dfrac{3}{4}\sqrt{\dfrac{1}{2}} - \dfrac{1}{4}\sqrt{\dfrac{1}{2}} = \left(\dfrac{3}{4} - \dfrac{1}{4}\right)\sqrt{\dfrac{1}{2}} =$

$= \dfrac{1}{2}\cdot\sqrt{\dfrac{1}{2}}\ \mathrm{m}^3 \ (\approx 0.35\ \mathrm{m}^3)$

(b) cylinder:



Want: maximize volume $V = \pi r^2 \cdot h$

Constraint. $\underbrace{2\pi r^2}_{\text{bottom+top}} + \underbrace{2\pi r \cdot h}_{\text{side}} = 3 \rightsquigarrow 2\pi r h = 3 - 2\pi r^2$

$h = \dfrac{3 - 2\pi r^2}{2\pi r}$

objective function $V = \pi \cdot r^2 \cdot \dfrac{3 - 2\pi r^2}{2\pi r} = \dfrac{3}{2}r - \pi \cdot r^3$

$V' = \dfrac{3}{2} - 3\pi r^2 = 0$

$3\pi r^2 = \dfrac{3}{2}$

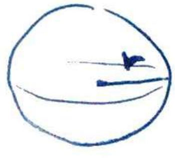
$\pi r^2 = \dfrac{1}{2}$

$r^2 = \dfrac{1}{2\pi}$ $\qquad r = \sqrt{\dfrac{1}{2\pi}}$

Max volume $V = \dfrac{3}{2}\cdot\sqrt{\dfrac{1}{2\pi}} - \pi\cdot\left(\sqrt{\dfrac{1}{2\pi}}\right)^3$

$= \dots = \dfrac{\sqrt{2\pi}}{2\pi} \ (\approx 0.4\ \mathrm{m}^3)$

(c) sphere (just for comparison): (← not an optimization problem only one sphere with surface area = 3 exists)



Surface area $= A = 4\pi r^2 = 3$

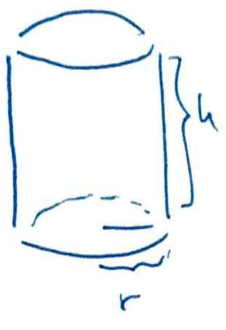
$\rightsquigarrow r^2 = \dfrac{3}{4\pi}$

$r = \sqrt{\dfrac{3}{4\pi}}$

Volume is $V = \dfrac{4}{3}\pi r^3 = \dfrac{4\pi}{3}\cdot\dfrac{3}{4\pi}\cdot\sqrt{\dfrac{3}{4\pi}}$

$= \sqrt{\dfrac{3}{4\pi}}\ \mathrm{m}^3 \ (\approx 0.49\ \mathrm{m}^3)$

Exercise: A soft drink company plans to make cyllindrical cans of volume exactly $300\,\mathrm{cm}^3$. The cost of aluminium to make the cans is \$0.001 per cm^2. What is the minimal possible cost of materials per can?



Want: minimize cost $C = 0.001 \underbrace{A}_{\text{surface area}} = 0.001\cdot\left(2\pi r^2 + 2\pi r h\right)$

Constraint: Volume $= 300$

$$\pi r^2 h = 300 \leadsto h = \frac{300}{\pi r^2}$$

$\leadsto$ obj. function $C = 0.001\cdot\left(2\pi r^2 + 2\pi r\cdot\frac{300}{\pi r^2}\right)$

$$= 0.002\pi r^2 + \frac{0.6}{r}$$

$$C' = 0.004\pi r - \frac{0.6}{r^2}$$

$$0.004\pi r - \frac{0.6}{r^2} = 0$$

$$0.004\pi r^3 - 0.6 = 0$$

$$0.004\pi r^3 = 0.6$$

$$4\pi r^3 = 600$$

$$r^3 = \frac{600}{4\pi} = \frac{150}{\pi}$$

$$r = \sqrt[3]{\frac{150}{\pi}}$$

$$\leadsto h = \frac{300}{\pi r^2} = \frac{300}{\pi\cdot\frac{\left(\sqrt[3]{150}\right)^2}{\left(\sqrt[3]{\pi}\right)^2}} = \frac{300}{\sqrt[3]{\pi}\cdot\left(\sqrt[3]{150}\right)^2}$$

(not necessary – no one asked for dimensions, just cost)

Minimal possible cost per can is

$$C = 0.002\cdot\pi\cdot\left(\sqrt[3]{\frac{150}{\pi}}\right)^2 + \frac{0.6}{\sqrt[3]{\frac{150}{\pi}}} = 0.002\cdot\sqrt[3]{\pi}\cdot\left(\sqrt[3]{150}\right)^2 + \frac{0.6\cdot\sqrt[3]{\pi}}{\sqrt[3]{150}}$$

$$\approx \$\,0.25$$