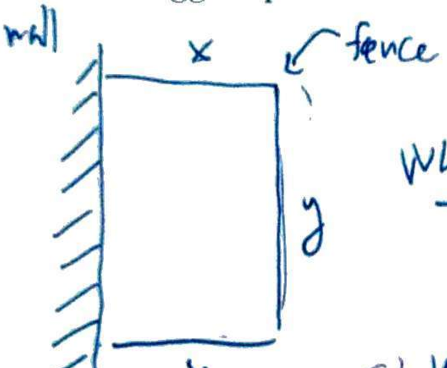
MA 16010 Lesson 24: Optimization I

Goal: Apply calculus in real(–ish)–life problems to optimize outcome: maximize something (profit, area, volume, ...) or minimize something (cost,...)

Example: A small rectangular garden should be enclosed by a wall on one side and a fence on the other sides. We have 20 m of fencing materials at our disposal. What are the dimensions of the garden so that its area is the biggest possible?



<u>Want</u>: maximize the area $A = xy$

<u>What can be said about x, y:</u>

$$2x + y = 20 \rightsquigarrow y = 20 - 2x$$

$\rightsquigarrow$ Want to maximize $A = x \cdot y = x(20-2x) = 20x - 2x^2$

$$A' = 20 - 4x$$

$$20 - 4x = 0$$

$$20 = 4x$$

$\underline{\underline{5 = x}}$ the only critical point.

<u>Dimensions of the garden:</u> $\boxed{\begin{array}{l} x = 5 \text{ m} \\ y = 20 - 2 \cdot 5 = 10 \text{ m} \end{array}}$

Optimization problems in steps:

1. draw a picture (if appropriate), set up variables
2. find the quantity to be optimized
3. find the constraints on the variables
4. Use $\boxed{2.}$ & $\boxed{3.}$ to obtain objective function = function of 1 var to be optimized
5. Take derivative of the obj. function, set it to $=0$ to find crit. points.
6. Typically: we get one crit. point, that's the one we want (no need for 1st/2nd derivative test). Compute all the other quantities we need.

Example: Find the pair of non–negative numbers such that their product is 25 and their sum is minimal possible.

1. x, y are the pair of numbers
2. Want: minimize ~~xy~~ $S = x + y$
3. $xy = 25 \quad x \geq 0, \ y \geq 0$
4. $y = \frac{25}{x} \rightsquigarrow \boxed{S = x + \frac{25}{x}} = x + 25x^{-1}$ ... want this to be minimized
5. $S' = 1 - \frac{25}{x^2}$

$1 - \frac{25}{x^2} = 0$

$1 = \frac{25}{x^2}$

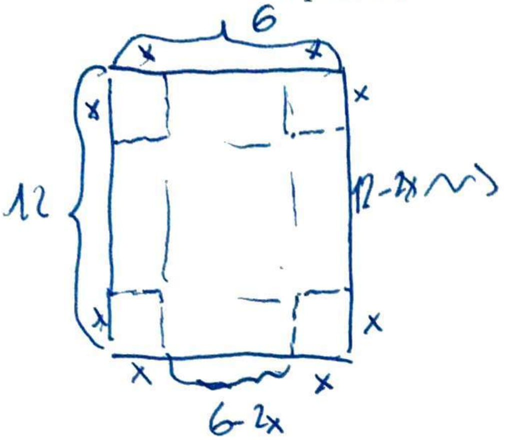
$x^2 = 25$

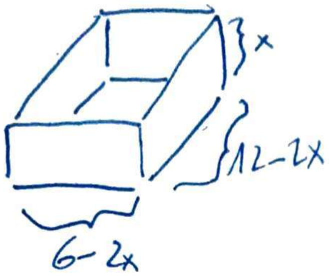
$x = \pm 5 \quad \ldots \quad x \geq 0$

$\to \underline{x = 5}$

6. $\boxed{\begin{array}{l} x = 5 \\ y = \frac{25}{x} = \frac{25}{5} = 5 \end{array}}$

Example: A piece of cardboard has dimensions 6 in $\times$ 12 in. A square is to be cut from each corner and the sides folded up to make an open-top box. What is the maximum possible volume of the box? Round your answer to three decimal places.





Want: maximize the volume

$$V = (6-2x)(12-2x)\cdot x =$$
$$= (72 - 24x - 12x + 4x^2)x =$$
$$= 72x - 36x^2 + 4x^3$$

constraints: $x \geq 0$

$$6 - 2x \geq 0 \qquad 12 - 2x \geq 0$$
$$6 \geq 2x \qquad 12 \geq 2x$$
$$x \leq 3 \qquad x \leq 6$$

$$\underline{0 \leq x \leq 3}$$

$$V' = 72 - 72x + 12x^2$$

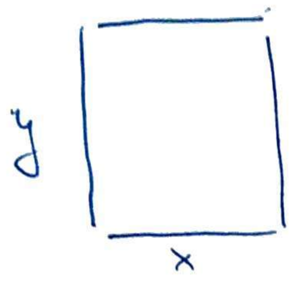
$$12x^2 - 72x + 72 = 0$$
$$x^2 - 6x + 6 = 0$$
$$x_{1,2} = \frac{6 \pm \sqrt{36 - 4\cdot 6}}{2} = \frac{6 \pm \sqrt{12}}{2} =$$
$$= 3 \pm \frac{2\sqrt{3}}{2} = \underline{3 \pm \sqrt{3}}$$

$$x \leq 3 \rightarrow \underline{x = 3 - \sqrt{3}}$$

$\rightarrow$ The maximal possible volume is

$$V = (6 - 2(3-\sqrt{3}))(12 - 2(3-\sqrt{3}))(3-\sqrt{3})$$
$$\approx \underline{41.569 \text{ in}^3}$$

Example: A carpenter is building a rectangular room with a fixed perimeter of N feet (where N is a fixed positive number). What are the dimensions of the largest room that can be built, and what is the room's area?



Want: maximize area $A = xy$

Constraints: $x \geq 0$, $y \geq 0$, and

(Perimeter =) $2x + 2y = N$

$\leadsto y = \frac{N-2x}{2} = \frac{N}{2} - x$

Obj. function $A = xy = x\left(\frac{N}{2} - x\right)$

$A = \frac{N}{2}x - x^2$

$A' = \frac{N}{2} - 2x$

$\frac{N}{2} - 2x = 0$

$2x = \frac{N}{2}$

$x = \frac{N}{4}$

Dimensions of the largest room are $\boxed{\begin{aligned} x &= \tfrac{N}{4} \text{ ft} \\ y &= \tfrac{N}{4} \text{ ft} \end{aligned}}$

and the room's area is $\boxed{A = xy = \frac{N^2}{16} \text{ ft}^2}$.