MA 16010 Lesson 21: Limits at ∞

Recall: The expression

$$\lim_{x\to c} f(x) = \infty$$

has the meaning:

"As x approaches c, the value $f(x)$ grows above any bound („approaches ∞")."

Example: $\lim_{x\to 0} \frac{1}{x^2} = \infty$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	100	10000	1000000	—	1000000	10000	100

Now we switch it around: The expression

$$\lim_{x\to\infty} f(x) = c$$

has the meaning:

"As x approaches ∞/grows beyond any bound, the value $f(x)$ approaches c."

Example: $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

x	100	10000	1000000	100000000	...
$f(x)$	0.1	0.01	0.001	0.0001	...

Similarly:

- We may also consider $\lim_{x\to-\infty} f(x)$: „as x approaches $-\infty$ / decreases ~~beyond~~ below any given bound."
- It may happen that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$ etc.

Limits at $\pm\infty$ of rational functions:

- We may use the same computational fules for limits as before:

$$\lim_{x\to\infty}\big(f(x)+g(x)\big) = \big(\lim_{x\to\infty} f(x)\big) + \big(\lim_{x\to\infty} g(x)\big),$$

$$\lim_{x\to\infty}\big(f(x)/g(x)\big) = \big(\lim_{x\to\infty} f(x)\big)/\big(\lim_{x\to\infty} g(x)\big), \quad \text{etc.}$$

- Useful observation: if c is a constant and a is a positive exponent, then

$$\lim_{x\to\infty}\frac{c}{x^a} = 0, \qquad \left[\text{Ex: } \lim_{x\to\infty}\frac{1}{x^{\frac{1}{2}}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0\right]$$

$$\lim_{x\to-\infty}\frac{c}{x^a} = 0, \quad \text{if it makes sense, e.g. if } a \text{ is positive integer}$$

Example:

$$\lim_{x\to-\infty}\left(\frac{3}{x}+\frac{x}{5}\right) = \lim_{x\to-\infty}\left(\frac{3}{x}\right) + \lim_{x\to-\infty}\left(\frac{x}{5}\right) = 0 + \text{“}\frac{-\infty}{5}\text{''} = -\infty$$

Example:

$$\lim_{x\to\infty}\frac{3x^3+7x^2-10x+5}{x^2+8x-6} = \lim_{x\to\infty}\frac{3x^3\left(1+\frac{7/3}{x}-\frac{10/3}{x^2}+\frac{5/3}{x^3}\right)}{x^2\left(1+8/x-\frac{6}{x^2}\right)} =$$

$$= \lim_{x\to\infty}\frac{3x^3}{x^2}\cdot\lim_{x\to\infty}\frac{1+\frac{7/3}{x}-\frac{10/3}{x^2}+\frac{5/3}{x^3}}{1+8/x-\frac{6}{x^2}} = \lim_{x\to\infty}\frac{3x^3}{x^2}\cdot 1 = \lim_{x\to\infty} 3x = \infty$$

General rule: When taking $\lim_{x\to\pm\infty}$ of a rational function, it is ok to disregard everything apart from the highest degree terms in numerator and denominator.

Example:

$$\lim_{x\to-\infty}\frac{5x^2+3-2x^4}{8x-3x^4+1} = \lim_{x\to-\infty}\frac{-2x^4}{-3x^4} = \lim_{x\to-\infty}\frac{-2}{-3} = \frac{2}{3}$$

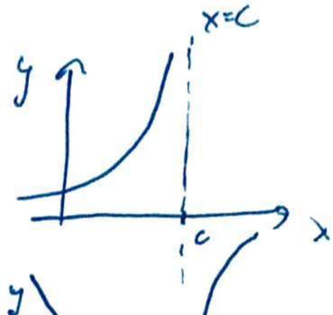
highest degree terms

Asymptotes. An asymptote of $f(x)$ is a line such that the graph of $f(x)$ tends to this line. Asymptotes can be

1. **Vertical:** $x = c$ is an asymptote for $f(x)$ if:

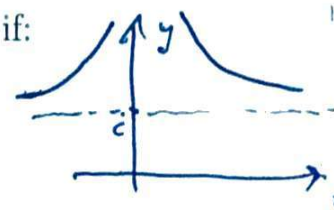
   $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$

   in practice: look for c when denominator is 0 (in the reduced form)



2. **Horizontal:** $y = c$ is an asymptote for $f(x)$ if:

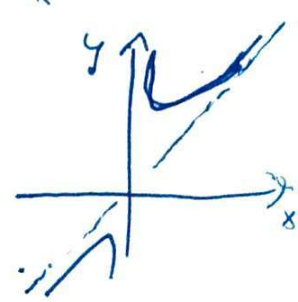
   $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$



3. **Slant:** $y = ax + b$ is an asymptote for $f(x)$ if:

   $\lim_{x \to \infty} \left(f(x) - (ax+b)\right) = 0$ or $\lim_{x \to -\infty} \left(f(x) - (ax+b)\right) = 0$

   in practice: for rat. function, degree of numerator has to be degree of denominator $+1$. Use long division to find $ax+b$



Example: Find horizontal, vertical, slant asymptotes of

$$f(x) = \frac{2x^2 - 7x - 6}{3x^2 - 12}.$$

1) Vertical: $3x^2 - 12 = 0$

$x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x = 2,\ x = -2$ (and neither makes the numerator $= 0$)

$\to$ vertical asymptotes $x = 2$, $x = -2$

2) horizontal:

$\lim_{x \to \infty} \frac{2x^2 - 7x - 6}{3x^2 - 12} = \lim_{x \to \infty} \frac{2x^2}{3x^2} = \lim_{x \to \infty} \frac{2}{3} = \frac{2}{3}$

$\lim_{x \to -\infty} \frac{2x^2 - 7x - 6}{3x^2 - 12} = \lim_{x \to -\infty} \frac{2x^2}{3x^2} = \lim_{x \to -\infty} \frac{2}{3} = \frac{2}{3}$

$\to$ horizontal asymptote $y = \frac{2}{3}$

3) slant: degree of numerator $= 2 =$ degree of denominator

$\to$ cannot have slant asymptote.

Example: Find horizontal, vertical, slant asymptotes of

$$f(x) = \frac{x^3 - 7x - 6}{x^2 + x - 2}.$$

1) <u>vertical:</u>

$x^2 + x - 2 = 0 \rightsquigarrow x = -2$ or $x = 1$

$(x+2)(x-1) = 0$

<u>But:</u> when $x = -2$, also numerator $= 0$

(in fact: $f(x) = \frac{(x^2-2x-3)(x+2)}{(x-1)(x+2)} = \frac{x^2-2x-3}{x-1}$ when $x \neq -2$)

$\to$ vertical asymptote only $x = 1$ (can check: $1^3 - 7 \cdot 1 - 6 \neq 0$)

2) <u>horizontal:</u>

$$\lim_{x\to\infty} \frac{x^3-7x-6}{x^2+x-2} = \lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty$$

$$\lim_{x\to-\infty} \frac{x^3-7x-6}{x^2+x-2} = \lim_{x\to-\infty} \frac{x^3}{x^2} = \lim_{x\to-\infty} x = -\infty$$

$\to$ no horizontal asymptotes

3) <u>slant:</u> degree of num. $= 3 = $ (degree of denom.) $+ 1$

$\rightsquigarrow$ we try long division:

$$\begin{array}{r} x - 1 \\ x^2+x-2 \overline{\big)\, x^3 - 7x - 6} \\ -\,(x^3 + x^2 - 2x) \\ \hline -x^2 - 5x - 6 \\ -\,(-x^2 - x + 2) \\ \hline -4x - 8 \end{array}$$

So ($f(x) = (x-1) + \frac{-4x-8}{x^2+x-2}$, hence) slant asymptote is $\underline{y = x - 1}$