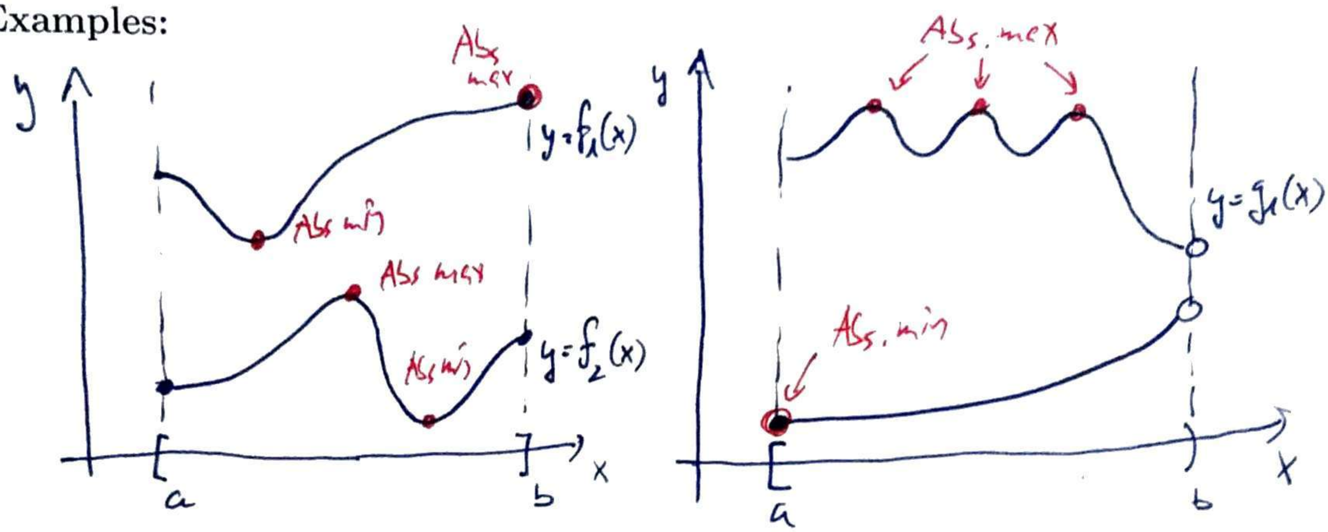
MA 16010 Lesson 20: Absolute extrema on an interval

The **absolute maximum** of $y = f(x)$ on an interval I is: the biggest value that f(x) attains on I (= f(c) such that f(c) ≥ f(x) for all x from I)

The **absolute minimum** of $y = f(x)$ on an interval I is: the smallest value that f(x) attains on I (= f(c) ... f(c) ≤ f(x) ...)

Examples:



I = [a, b]

for $f_1(x)$: Abs max at x=b → right endpoint
Abs min at a rel. min inside I → at a critical point

for $f_2(x)$: Abs max (min) at a rel. max (min) inside I → at crit. points

for $g_1(x)$: Abs. max at (three) rel. maxima → at crit. pts
Abs min DNE!

for $g_2(x)$: Abs min at x=a — left endpoint
Abs max DNE!

Observations:

- Abs min/max need not exist. If they do, they can be attained at multiple x's
- If abs max/min is attained at x, then x is either an endpoint (a/b) of I, or x is a critical point inside I.

Absolute extrema on a closed (and bounded) interval $I = [a,b]$ $a \leq b$ numbers

Fact: In this case, abs max & min exist (if f is continuous)

How to find them:

1. Find all the critical points of f in (a,b)
2. Evaluate f at all the crit. points, and also at the endpoints a, b
3. Abs. max = the biggest value from $\boxed{2.}$
   Abs. min = the smallest value from $\boxed{2.}$

Exercise: Find the abs. maximum and the abs. minimum of the function

$$f(x) = x^4 - 18x^2 + 5$$

on the interval $[-4, 6]$.

1. crit. points:

$f'(x) = 4x^3 - 36x$

$4x^3 - 36x = 0$

$x^3 - 9x = 0$

$x(x^2 - 9) = 0$

$x(x-3)(x+3) = 0$

$\rightsquigarrow$ crit. points $x_1 = 0$

$x_2 = 3$

$x_3 = -3$

2. evaluate:

$f(0) = 5$

$f(3) = 3^4 - 18 \cdot 3^2 + 5 = -81 + 5 = -76$

$f(-3) = (-3)^4 - 18 \cdot (-3)^2 + 5 = -81 + 5 = -76$

$f(-4) = (-4)^4 - 18 \cdot (-4)^2 + 5 = -32 + 5 = -27$

$f(6) = 6^4 - 18 \cdot 6^2 + 5 = 653$

3. conclusion

Abs. max: $(x,y) = (6, 653)$

Abs. min: $(x,y) = (3, -76)$, $(x,y) = (-3, -76)$

Exercise: Find the abs. maximum and the abs. minimum of the function

$$f(x) = 2x^3e^x + 7$$

on the interval $[-4, -2]$.

1. critical points:

$f'(x) = 6x^2e^x + 2x^3e^x = (6x^2+2x^3)e^x$

$(6x^2+2x^3)e^x = 0$

e^x never $=0$

$6x^2+2x^3 = 0$

$x^2(6+2x) = 0$

$x = 0$ does not count, it is outside of $I = [-4,-2]$!

$6+2x = 0$

$2x = -6$

$x = -3$

2. evaluate:

$f(-3) = 2\cdot(-3)^3\cdot e^{-3}+7 = 7-54e^{-3} \approx 4.311$

$f(-4) = 2\cdot(-4)^3e^{-4}+7 = 7-128e^{-4} \approx 4.656$

$f(-2) = 2\cdot(-2)^3e^{-2}+7 = 7-16e^{-2} \approx 4.835$

3. conclusion

Abs. min: $(x,y) = (-3, 7-54e^{-3})$

Abs. max: $(x,y) = (-2, 7-16e^{-2})$

Exercise: Find the abs. maximum and the abs. minimum of the function

$$f(x) = \frac{2}{2x^2+3}$$

on the interval $[-1, 1]$.

1. crit. pts:

$f'(x) = \frac{0(2x^2+3)-(4x)\cdot 2}{(2x^2+3)^2} = \frac{-8x}{(2x^2+3)^2}$

$\frac{-8x}{(2x^2+3)^2} = 0$

$-8x = 0$

$x = 0$

2. evaluate

$f(0) = \frac{2}{0+3} = \frac{2}{3}$

$f(-1) = \frac{2}{2\cdot(-1)^2+3} = \frac{2}{2+3} = \frac{2}{5}$

$f(1) = \frac{2}{2\cdot 1^2+3} = \frac{2}{2+3} = \frac{2}{5}$

3. conclusion:

Abs. max: $(x,y) = (0, \frac{2}{3})$

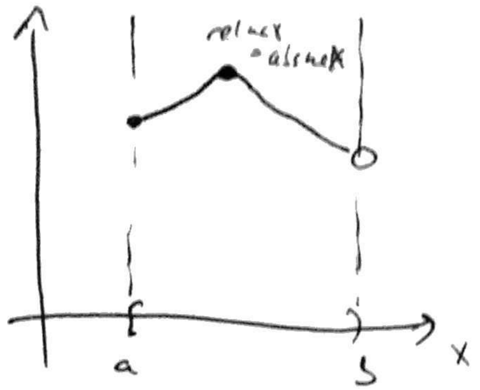
Abs. min: $(x,y) = (-1, \frac{2}{5})$

$(x,y) = (1, \frac{2}{5})$

Absolute extrema on a general (bounded) interval

We consider the **special case:** only one critical point in the interior of I

In this case: if the crit. point is a relative min/max, then it is also the absolute min/max!



⇝ once we find just one crit. point and see that it's a rel. extreme, it's the abs. extreme.

Exercise: (If it exists,) find the abs. minimum of the function

$$f(x) = x^3 - 3x + 2$$

on the interval $[0, 2)$.

1. crit. points:

$f'(x) = 3x^2 - 3$

$3x^2 - 3 = 0$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

only $x = 1$ is in $[0,2)$

→ only one crit. point.

2. Is the crit. point rel. extreme?

Can try first/second derivative test

2nd derivative test:

$f''(x) = 6x$

$f''(1) = 6 > 0 \Rightarrow$ rel. minimum

3. The unique crit. point is a rel. minimum

→ it's the absolute minimum

So, abs. min

$(x,y) = (1, f(1)) = (1, 0)$