MA 16010 Lesson 2: Limits Numerically

Limits.

Example. The function

$$f(x) = \frac{x^2-4}{x-2} \quad (= x+2 \text{ except when } x=2)$$

is not defined at $x = 2$: $f(2) = \frac{2^2-4}{2-2} = 0/0$ ... not defined!

We still wish to understand how the function behaves at least *near* $x = 2$. Let us list some values of $f(x)$ near $x = 2$:

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	—	4.001	4.01	4.1

We observe that as x approaches 2, the value $f(x)$ approaches 4.

We say that 4 is the *limit of* $f(x) = \frac{x^2-4}{x-2}$ *as* x *approaches* 2, also written as:

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$$

In general: We say that L is the limit of $f(x)$ as x approaches (a given number) c if: as x approaches c, the value $f(x)$ approaches L.

We write this fact as

$$\lim_{x \to c} f(x) = L \ .$$

Infinite limits: We can have $L = \infty$ or $L = -\infty$ in the above.

- $\lim_{x\to c} f(x) = \infty$ means: As x approaches c, the value $f(x)$ exceeds any bound ("$f(x)$ approaches ∞")
- $\lim_{x\to c} f(x) = -\infty$ means: As x approaches c, the value $f(x)$ gets smaller than any given bound ("$f(x)$ approaches $-\infty$

Exercise: List the indicated values, rounded to ~~4~~ 5 decimal places, and determine the indicated limit:

$$f(x) = \frac{\sin(x)}{x}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.99833	0.99998	1.00000	—	1.00000	0.99998	0.99833

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Exercise: List the indicated values, rounded to 4 decimal places, and determine the indicated limit:

$$f(x) = 2 + \frac{4}{(x+3)^2}$$

x	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9
$f(x)$	402	40 002	4000 002	—	4000 002	40 002	402

$$\lim_{x \to -3} \left(2 + \frac{4}{(x+3)^2}\right) = +\infty$$

One–sided limits.
Example. Consider the function

$$f(x) = \frac{x^2}{2x-2}$$

and its behaviour near $x = 1$.

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	-4.05	-49.005	-499.0005	—	501.0005	51.005	6.05

Does $\lim_{x\to 1} f(x)$ exist? NO (when x approaches x on the left/right, the behavior of f(x) is very different!)

What can be said:

- As x approaches 1 *from the left / from below*, $f(x)$ approaches $-\infty$.

  We say that the left–sided limit of $f(x)$ as x approaches 1 (from the left) is equal to $-\infty$. We also write: $\lim_{x\to 1^-} \frac{x^2}{2x-2} = -\infty$.

- As x approaches 1 *from the right / from above*, $f(x)$ approaches $+\infty$.

  We say that the right–sided limit of $f(x)$ as x approaches 1 (from the right ~~left~~) is equal to $+\infty$. We also write: $\lim_{x\to 1^+} \frac{x^2}{2x-2} = +\infty$.

In general: We say that L is the limit of $f(x)$ as x approaches (a given number) c from the left / from the right if $f(x)$ tends to L as x approaches c from the left / from the right. We write this also as

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L, \quad \text{resp.}$$

Relation to "both–sided limits": $\lim_{x\to c} f(x)$ exists and equals L if and only if: $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$.

Exercise: List the indicated values, rounded to 4 decimal places, and determine the indicated limit:

$$f(x) = \frac{x+2}{x^2-2x-8} \quad \left(= \frac{1}{x-4} \text{ when } x \neq 4 \right)$$

x	4	4.0001	4.001	4.01	4.1
$f(x)$	—	10 000	1000	100	10

$\lim_{x \to 4^+} \frac{x+2}{x^2-2x-8} = \infty$

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Exercise: List the indicated values, rounded to 4 decimal places, and determine the indicated limits:

$$f(x) = \frac{|x|}{x}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-1	-1	-1	—	1	1	1

$\lim_{x \to 0^-} \frac{|x|}{x} = -1$

$\lim_{x \to 0^+} \frac{|x|}{x} = 1$

$\lim_{x \to 0} \frac{|x|}{x} =$ DNE (does not exist: $\lim_{x \to 0^-} \frac{|x|}{x}$, $\lim_{x \to 0^+} \frac{|x|}{x}$ do not agree!)