# MA 16010 Lesson 19: Concavity, inflection pts, 2nd derivative test

**Recall:**

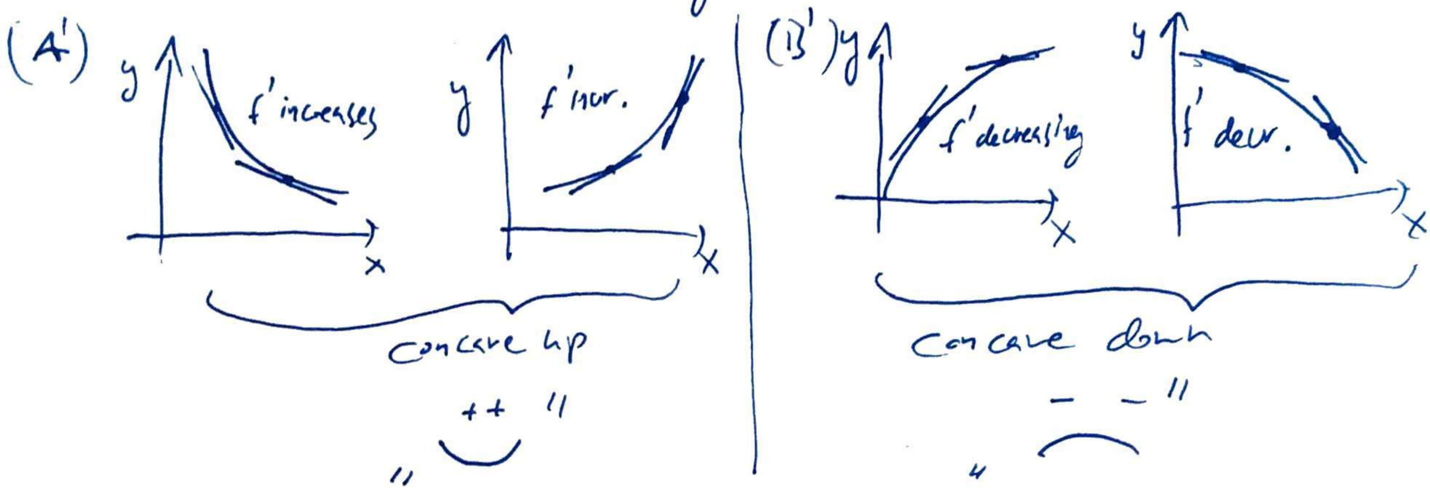
(A) If $f'(x) > 0$ on an interval $I$, then $f$ is <u>increasing</u> in $I$.

(B) If $f'(x) < 0$ on an interval $I$, then $f$ is <u>decreasing</u> in $I$.

Now let's go one step further:

(A') $f''(x) > 0$ on $I \Rightarrow$ <u>$f'$ increasing</u>; then $f$ is <u>concave up</u> on $I$.

(B') $f''(x) < 0$ on $I \Rightarrow$ <u>$f'$ decreasing</u>; then $f$ is <u>concave down</u> on $I$.




**Example:** Find the largest intervals where $f(x) = x^3 - 3x^2 + 7x + 1$ is concave up and concave down. ← we find intervals where $f'$ is increasing or decreasing

$f'(x) = 3x^2 - 6x + 7$

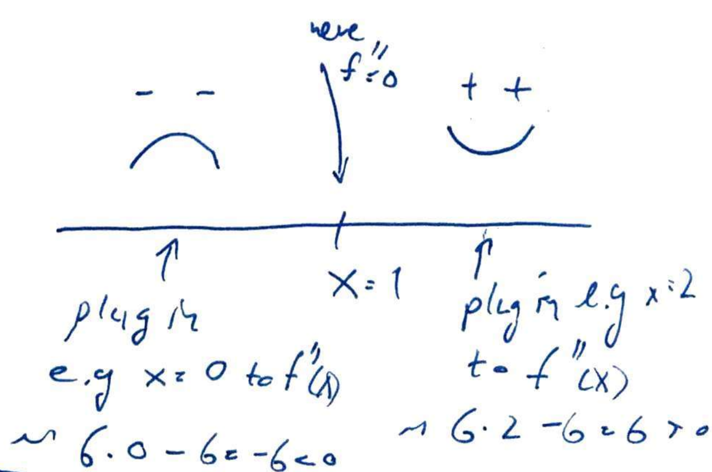
$f''(x) = 6x - 6$

$f''(x) = 0$

$6x - 6 = 0$

$6x = 6$

$x = 1$

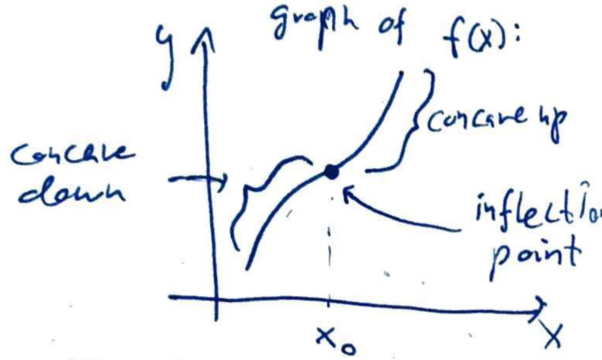


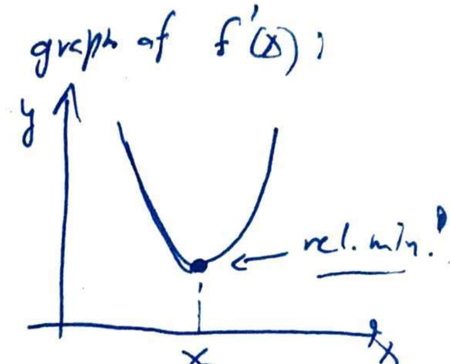

Conclusion: Concave down on $(-\infty, 1)$

Concave up on $(1, \infty)$

A point $(x, y)$ where $y = f(x)$ changes from concave up to concave down or vice versa is called an inflection point.

To find such points is to find relative extrema of $f'(x)$ !




**Exercise:** Find the largest intervals on which the function

$$f(x) = \frac{x^4}{3} + \frac{2}{3}x^3 - 4x^2 + x + 1$$

is concave up or concave down, and find the inflection points.

$f'(x) = \frac{4}{3}x^3 + 2x^2 - 8x + 1$

$f''(x) = 4x^2 + 4x - 8$

1. "crit. pts" (of $f'(x)$):

$$4x^2 + 4x - 8 = 0$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x_1 = 1, \quad x_2 = -2$$

2. Intervals where $f'' > 0$, $f'' < 0$:


plug in $-3$: $4 \cdot 9 - 4 \cdot 3 - 8 = 24 - 8 = 16 > 0$

plug in 0 to $f''(x)$, get $-8 < 0$

plug in 2 $\leadsto 16 + 8 - 8 = 16 > 0$

3. Conclusions:

- Concave up on $(-\infty, -2), (1, \infty)$
- Concave down on $(-2, 1)$
- inflection pts:

at $(x, y) = (-2, f(-2)) = (-2, -17)$

$(x, y) = (1, f(1)) = (1, -1)$

**Summary - inflection points.**

1. Find all $x$'s where $f''(x)=0$, or where $f''(x)$ DNE.
2. On each interval between the points from $\boxed{1.}$ determine whether $f''>0$, or $f''<0$ (so whether $f$ is concave up/down)
3. Inflection pts = the points from $\boxed{1.}$ where $f$ switches concavity.

**Exercise:** Find the largest intervals on which the function

$$f(x) = 5\ln(x^2+4)$$

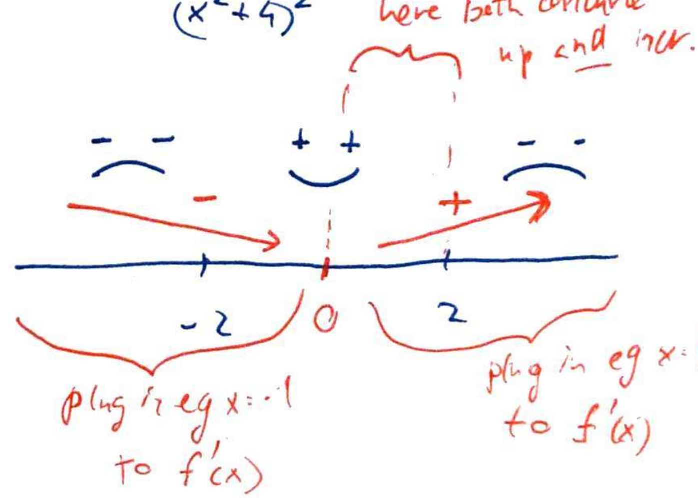
is: (a) concave up or concave down, and find the inflection points.
(b) concave up and increasing (at the same time).

$$f'(x) = 5\cdot\frac{1}{x^2+4}\cdot 2x = \frac{10x}{x^2+4}$$

$$f''(x) = \frac{10(x^2+4)-2x\cdot 10x}{(x^2+4)^2} = -\frac{10x^2-40}{(x^2+4)^2}$$

(a) $f''(x)=0 \quad\ldots\quad -\frac{10x^2-40}{(x^2+4)^2}=0$

$$10x^2-40=0$$
$$x^2-4=0$$
$$x^2=4$$
$$\underline{x=\pm 2}$$




$\boxed{(b)}$ $f'(x)=0 \ldots \quad \frac{10x}{x^2+4}=0$

$$10x=0$$
$$\underline{x=0}$$

<u>we have</u>

(a) concave up on $(-2,2)$ //
concave down on $(-\infty,-2)$, $(2,\infty)$

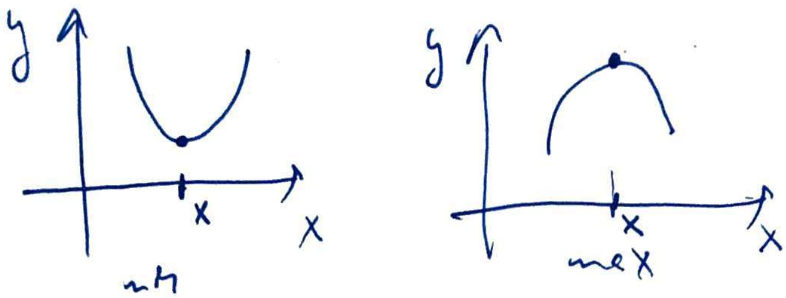
(b) increasing on $(0,\infty)$

→ concave up and increasing on $\underline{(0,2)}$

We may use concavity in finding relative extrema. If $x$ is a point of:

(a) rel. max., then $f$ is typically concave down around it, so we expect $f''(x) < 0$.

(b) rel. min., then $f$ is typically concave up around it, so we expect $f''(x) > 0$.




**Second derivative test:** Let $x$ be a critical point of $y = f(x)$.

1. If $f''(x) < 0$, then x is a point of rel. maxima of f.
2. If $f''(x) > 0$, then x is a point of rel. minima of f.
3. In other cases, the test is inconclusive!

**Exercise:** Find the rel. extrema of $f(x) = \frac{2}{3}x^3 - x^2 - 12x + 5$.

1. critical points:

$f'(x) = 2x^2 - 2x - 12$

$2x^2 - 2x - 12 = 0$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3$ or $x = -2$

2. second derivative test:

$f''(x) = 4x - 2$

$f''(-2) = 4 \cdot (-2) - 2 = -10 < 0$

$\rightarrow$ rel. maximum

$f''(3) = 4 \cdot 3 - 2 = 10 > 0$

$\rightarrow$ rel. minimum

3. conclusion:

rel. minimum at $(x, y) = (3, f(3)) = (3, -22)$

rel. maximum at $(x, y) = (-2, f(-2)) = (-2, \frac{59}{3})$