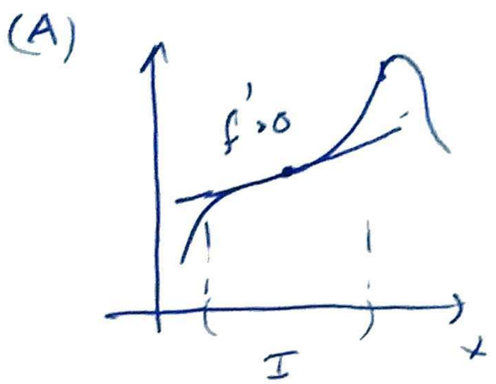
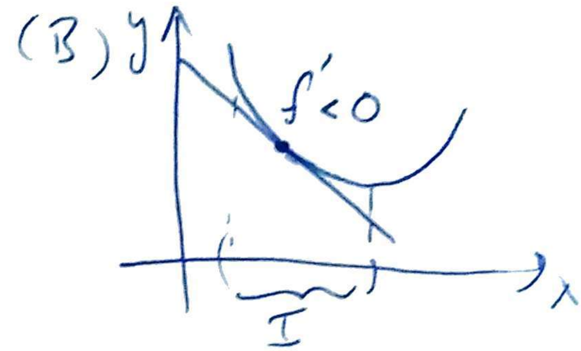
MA 16010 Lesson 18: Incrasing & decreasing, first derivative test

Observation: Recall that the derivative of a function $y = f(x)$ has the meaning of *rate of change of f*. Therefore:

(A) If $f'(x) > 0$ on an interval I, then f is increasing in I.

(B) If $f'(x) < 0$ on an interval I, then f is decreasing in I.

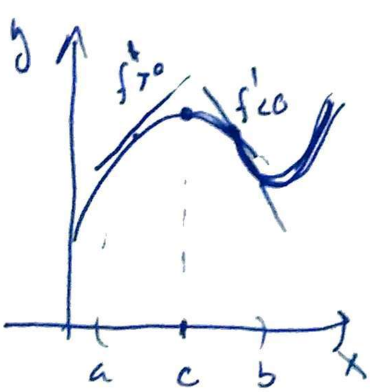


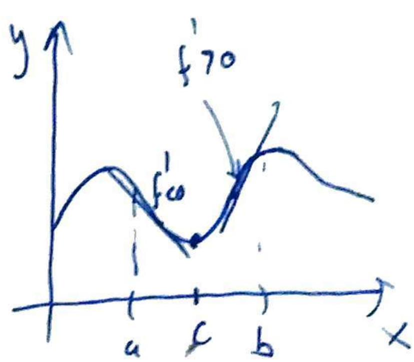


Application for relative extrema:

How to tell if a critical point is rel. maximum/rel. minimum?

- If c is the point of rel. maximum of f, then f is increasing on some interval (a, c), decreasing on some interval (c, b).
- If c is the point of rel. minimum of f, then f is decreasing on some interval (a, c), increasing on some interval (c, b).

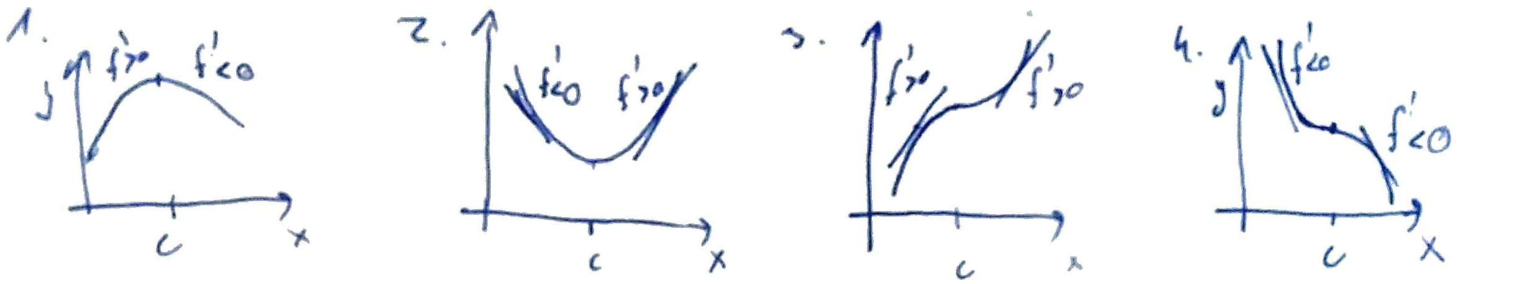




Idea: Based on where $f'(x) < 0$ and where $f'(x) > 0$, determine which type of rel. extreme we are dealing with.

First derivative test: Given a critical point c of $f(x)$:

	if ...	then	
1.	$f'(x) > 0$ on the left, $f'(x) < 0$ on the right,	rel. maximum	at c
2.	$f'(x) < 0$ on the left, $f'(x) > 0$ on the right,	rel minimum	at c
3.	$f'(x) > 0$ on the left, $f'(x) > 0$ on the right,	neither	at c
4.	$f'(x) < 0$ on the left, $f'(x) < 0$ on the right,	neither	at c



Strategy for relative extrema:

1. Find all the critical points (by $f'(x)=0$, and where $f'(x)$ undefined)
2. ~~Determi~~ On all intervals in between the crit. pts, find if $f'(x)>0$ or $f'(x)<0$
3. Find rel. maxima/minima based on First Derivative test.

Exercise: Find the rel. extrema of $f(x) = -2x^3 + 3x^2 + 12x + 5$.

1. crit. pts

$f'(x) = -6x^2 + 6x + 12$

$-6x^2 + 6x + 12 = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

crit. pts are $\boxed{x_1 = -1, \; x_2 = 2}$

2. Determine $f'>0$, $f'<0$:

$\searrow$ $\nearrow_+$ $\searrow$

(plug in e.g. -2) $\;-1\;$ (plug in 0) $\;2\;$ (plug in e.g. 3)

3. First derivative test:

$x_1 = -1$ -- ~~local~~ rel. minimum $(-1, -2)$

$y = -2\cdot(-1) + 3\cdot(-1)^2 + 12\cdot(-1) + 5$

$= -2$

$x_2 = 2$ -- rel. maximum $(2, 25)$

$y = -2\cdot 2^3 + 3\cdot 2^2 + 12\cdot 2 + 5 = 25$

Exercise: The derivative of a function $f(x)$ is $f'(x) = e^{3x}(x^3 + x^2 - 6x)$. Find the points of relative minima and maxima of $f(x)$.

1. crit pts: $f'(x) = 0$ ← this is what we are given!

$\underbrace{e^{3x}}_{\neq 0 \text{ ever}}(x^3+x^2-6x) = 0$

$\to x^3+x^2-6x=0$

$x(x^2+x-6)=0$

$x=0$ or $x^2+x-6=0$

$(x+3)(x-2)=0$

$x=-3,\ x=2$

$\to$ crit pts $x_1=-3,\ x_2=0,\ x_3=2$

2. Determine $f'(x)>0$, $f'(x)<0$:

$-$ | -3 | $+$ | 0 | $-$ | 2 | $+$

(plug in e.g. -4) -3 (plug in e.g. -1) 0 (plug in e.g. 1) 2 (plug in 3)

rel. minimum at $x=-3$, $x=2$

rel maximum at $x=0$

(no way to find y-value.)

Exercise: The critical points of $f(x) = 2\cos(2x) + 2x$ on $(0, 2\pi)$ are:

$$x = \frac{\pi}{12}, \quad x = \frac{5\pi}{12}, \quad x = \frac{13\pi}{12}, \quad x = \frac{17\pi}{12}.$$

Find the x–values in $(0, 2\pi)$ at which $f(x)$ has a relative maximum.

~~Here~~ 1. Done!

2. Need to determine $f'(x)>0$, $f'(x)<0$

$\to$ need $f'(x)$.

$f'(x) = -2\sin(2x)\cdot 2 + 2 = -4\sin(2x)+2$

$+$ | $-$ | $+$ | $-$ | $+$

0 (a) $\frac{\pi}{12}$ (b) $\frac{5\pi}{12}$ (c) $\frac{13\pi}{12}$ (d) $\frac{17\pi}{12}$ (e) 2π

(a) $f'\left(\frac{\pi}{24}\right)>0$ (use calculator)

(b) $f'\left(\frac{\pi}{4}\right) = -4\cdot\sin\left(\frac{\pi}{2}\right)+2 = -2<0$

(c) $f'\left(\frac{\pi}{2}\right) = -4\sin(\pi)+2 = 2>0$

(d) $f'\left(\frac{15\pi}{12}\right) = f'\left(\frac{5\pi}{4}\right) = -4\sin\left(\frac{5\pi}{2}\right)+2 = -2<0$

(e) $f'\left(\frac{21\pi}{12}\right) = f'\left(\frac{7\pi}{4}\right) = -4\sin\left(\frac{7\pi}{2}\right)+2 = 6>0$

Conclusion:

Rel. max. at $x=\frac{\pi}{12}$, $x=\frac{13\pi}{12}$

(rel. min. at $x=\frac{5\pi}{12}$, $x=\frac{17\pi}{12}$)