MA 16010 Lesson 17: Relative extrema, critical numbers

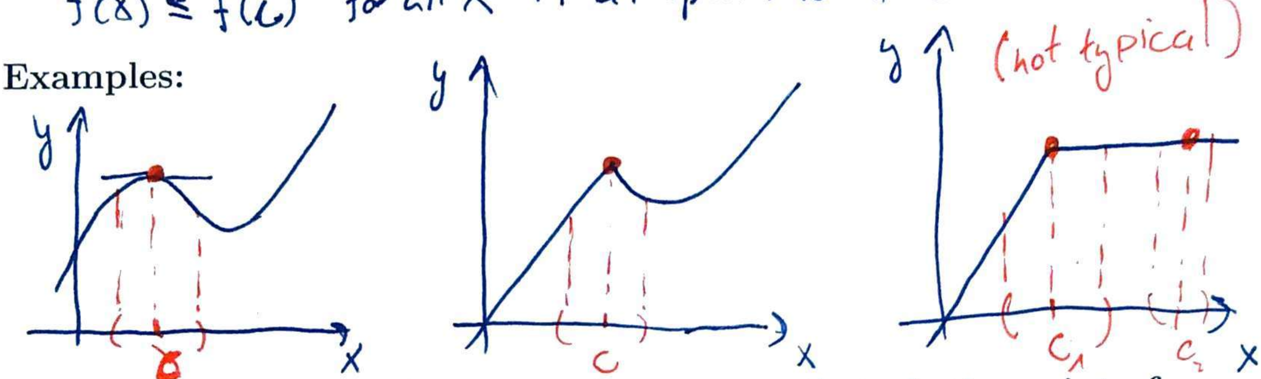
Given a function $y = f(x)$, we are often interested in its **maximal value** (e.g. "maximize profit") or its **minimal value** ("minimize costs"), if such values exist.

Today: We focus on **relative** maxima/minima.

- For a function $y = f(x)$ and a number c, we say that c **is the point of rel. maximum of** f / $f(c)$ **is a relative maximum** if:

$f(x) \leq f(c)$ for all x in an open interval I around c

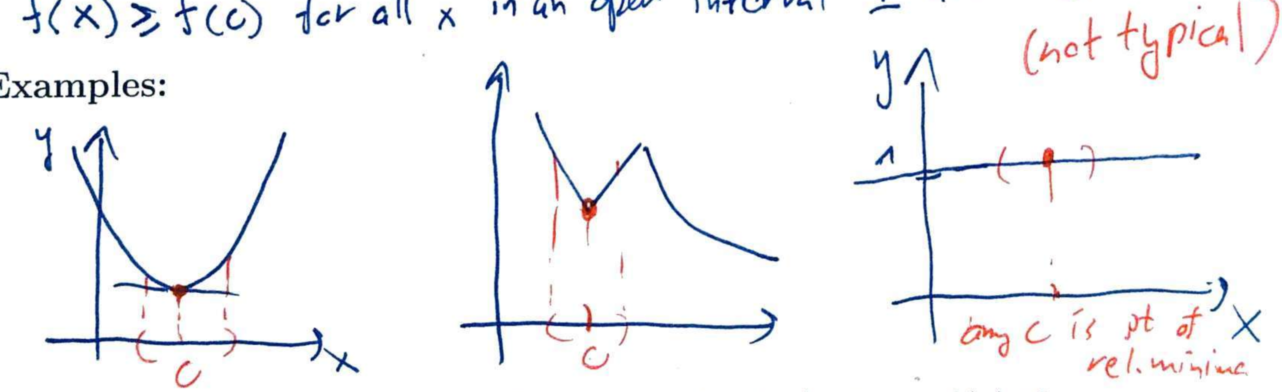
Examples:



- For a function $y = f(x)$ and a number c, we say that c **is the point of rel. minimum of** f / $f(c)$ **is a relative minimum** if:

$f(x) \geq f(c)$ for all x in an open interval I around c

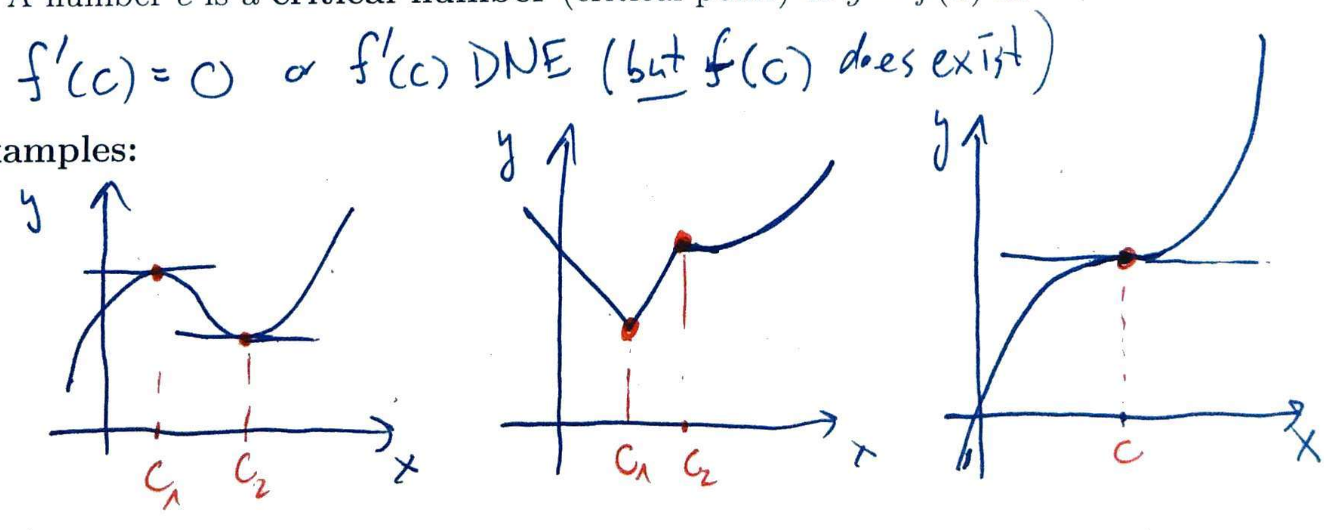
Examples:



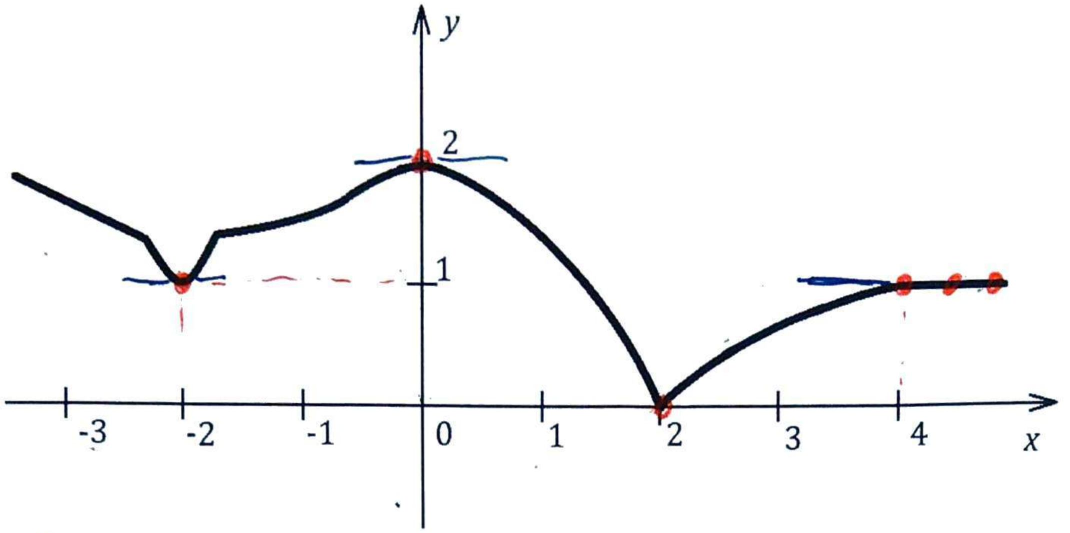
- A number c is a **critical number** (critical point) of $y = f(x)$ if:

$f'(c) = 0$ or $f'(c)$ DNE (but $f(c)$ does exist)

Examples:



Exercise: Find all relative extrema c, and describe $f'(c)$ at these points.



rel. minima

$(x,y) = (-2,1) \quad f'(-2)=0 \quad \left(+ (x,1), \text{ for } x>4, \; f'(x)=0 \right)$

$(x,y) = (2,0) \quad f'(2)$ DNE

rel maxima

$(x,y) = (0,2), \quad f'(0)=0 \quad \left(+ (x,1) \text{ for } x>4, \; f'(x)=0 \right)$

$(x,y) = (4,1), \quad f'(4)=0$

How to find relative extrema "analytically"?

Key observation: Relative minima, maxima are critical points → we find the critical points instead.

(**Warning:** Not every crit. point is a point of rel. min. or rel. max.!)

How to find critical points:

- Compute $f'(x)$
- Find all x s.t. $f'(x)=0$ ⇝ those will be crit. points,
- Find all x s.t. $f'(x)$ is not defined, if $f(x)$ is defined in these points, those are also the crit. pts.

Exercise: Find the critical numbers for the following functions.

(a) $y = x^3 - 24x + 15$:

$$y' = 3x^2 - 24$$

$$3x^2 - 24 = 0$$

$$x^2 - 8 = 0$$

$$x^2 = 8$$

$$x = \pm\sqrt{8} = \pm 2\sqrt{2}$$

crit. pts. are $\boxed{\begin{array}{l} x_1 = -2\sqrt{2} \\ x_2 = 2\sqrt{2} \end{array}}$

(b) $y = 2x^3 + 6x^2 + 6x + 1$:

$$y' = 6x^2 + 12x + 6$$

$$6x^2 + 12x + 6 = 0$$

$$x^2 + 2x + 1 = 0$$

$$(x+1)(x+1) = 0$$

$$\boxed{x = -1}$$

OR:

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1}}{2} = \frac{-2 \pm \sqrt{0}}{2} = \frac{-2}{2} = -1$$

critical numbers are $x = -1$

(c) $y = x^4 - 4x^3 + 4x^2 - 5$:

$$y' = 4x^3 - 12x^2 + 8x$$

$$4x^3 - 12x^2 + 8x = 0$$

$$x^3 - 3x^2 + 2x = 0$$

$$x(x^2 - 3x + 2) = 0$$

$x = 0$

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0$$

$x = 2$

$x = 1$

critical numbers are

$$\boxed{\begin{array}{l} x_1 = 0 \\ x_2 = 1 \\ x_3 = 2 \end{array}}$$

Exercise: Find the critical numbers for the following functions.

(a) $y = x^2 - \frac{3}{x^2} = x^{2} - 3x^{-2}$

$f'(x)$ is not defined at $x=0$
$f(x)$ is also not defined at $x=0$
$\rightsquigarrow$ no crit. pts this way

$y' = 2x - 3\cdot(-2)\cdot x^{-3}$
$= 2x + 6x^{-3} = 2x + \frac{6}{x^3}$

$2x + \frac{6}{x^3} = 0 \quad | \cdot x^3$

$2x^4 + 6 = 0$

$2x^4 = -6$

$\underbrace{x^4}_{\geq 0} = \underbrace{-3}_{\text{negative}}$ no solutions

$\boxed{\text{no critical points}}$

(b) $y = \underbrace{3x^3}\underbrace{e^{2x+1}}$:

$y' = 9x^2 \cdot e^{2x+1} + 3x^3 \cdot e^{2x+1} \cdot (2)$

$y' = 9x^2 \cdot e^{2x+1} + 6x^3 e^{2x+1}$

$= (9x^2 + 6x^3)e^{2x+1}$

$(9x^2 + 6x^3)\underbrace{e^{2x+1}}_{\text{always } \neq 0} = 0$

$9x^2 + 6x^3 = 0$

$x^2(9 + 6x) = 0$

$x = 0 \qquad 9 + 6x = 0$

$6x = -9$

$x = -9/6 = -3/2$

critical pts
$x_1 = 0$
$x_2 = -\frac{3}{2}$

(c) $y = \sin(4x) - 2x$, only in the interval $(0, \pi)$:

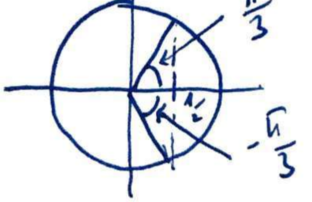
$y' = \cos(4x)\cdot 4 - 2$

$\rightsquigarrow 4\cos(4x) - 2 = 0$

$\cos(4x) - \frac{1}{2} = 0$

$\cos(4x) = \frac{1}{2}$

Say $u := 4x \rightsquigarrow \cos(u) = \frac{1}{2}$:



get $u = \frac{\pi}{3} + 2\pi\cdot k$, k integer, (1) or

$u = -\frac{\pi}{3} + 2\pi k$, k integer (2)

know: $x \in (0, \pi) \longrightarrow u = 4x \in (0, 4\pi)$

$\rightsquigarrow u = \underbrace{\frac{\pi}{3}, \frac{\pi}{3} + 2\pi}_{\text{from (1)}}, \underbrace{-\frac{\pi}{3} + 2\pi, -\frac{\pi}{3} + 4\pi}_{\text{from (2)}}$

$x = \frac{u}{4} = \frac{\pi}{12}, \frac{\pi}{12} + \frac{\pi}{2}, -\frac{\pi}{12} + \frac{\pi}{2}, -\frac{\pi}{12} + \pi$

$\boxed{\text{crit. points } x = \frac{\pi}{12},\ x = \frac{7\pi}{12},\ x = \frac{5\pi}{12},\ x = \frac{11\pi}{12}}$