MA 16010 Lesson 15: Related rates I

Before last time: Functions in explicit form $y = f(x)$, such as

$$y = x^2 .$$

Last time: Functions in implicit form. For example,

$$y^2 + 3xy = x^2 + y,$$

but $y = y(x)$ is "secretly" a function of x.

Today: We consider again general equations such as

$$y^2 + 3xy = x^2 + y,$$

but this time *both* $x = x(t)$ *and* $y = y(t)$ are "secretly" a function of a third variable t.

Similarly as with implicit derivatives, we can then relate the derivatives $x' = \frac{dx}{dt}$ and $y' = \frac{dy}{dt}$. (Their *rates of change* will be *related.*)

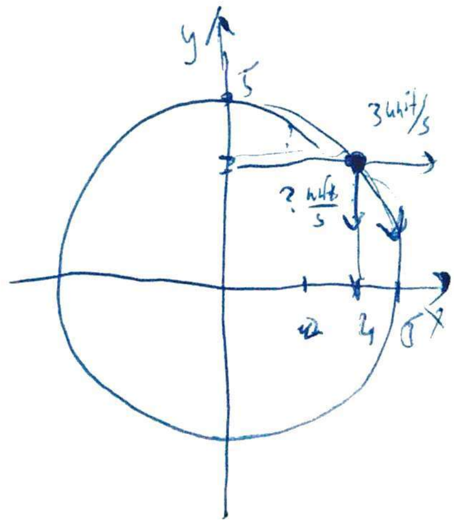
Example: A particle is moving on a circle of radius 5 centered at the origin. Its position (x, y) in the xy–plane therefore always satisfies the equation

$$x^2+y^2=25$$

When it passes through the point $(4, 3)$ x–coordinate changes at the rate $\frac{dx}{dt} = 3$ (units/second). What is the rate of change of the y–coordinate, $\frac{dy}{dt}$?

$$x^2+y^2=25 \quad \Big| \ \frac{d}{dt}[\ldots]$$

$$2x\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt} = 0$$



know: • we are at the point $(4, 3)$

$\rightarrow$ $x = 4$, $y = 3$

• at that moment, $\frac{dx}{dt} = 3$ unit/s

$\rightsquigarrow$ $2\cdot 4\cdot 3 + 2\cdot 3\cdot\frac{dy}{dt} = 0$

$6\frac{dy}{dt} = -24$ $\rightsquigarrow$ $\boxed{\frac{dy}{dt} = -4 \text{ unit}/s}$

Summary (finding related rates): Take the derivative on both sides of the equation with respect to t. This time, use the chain rule/implicit differentiation for both $x = x(t)$ and $y = y(t)$. That is,

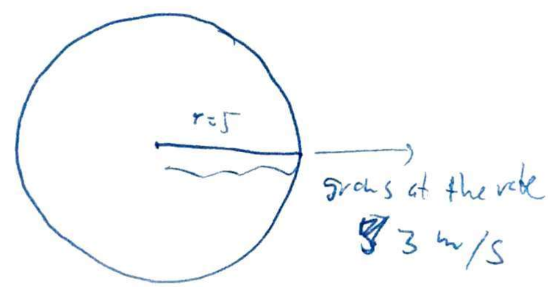
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[g(x)\right] = g'(x) \cdot \frac{dx}{dt} \left(= g'(x) \cdot x'\right), \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[h(y)\right] = h'(y) \cdot \frac{dy}{dt} \left(= h'(y)\, y'\right)$$

In the end, one gets an equation involving $x, y, x' = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $y' = \frac{\mathrm{d}y}{\mathrm{d}t}$. Plug in the ones that you know, and solve for the one you are trying to find.

Exercise: A radius of a circle is growing at the rate of 3 meters per second at the time when its radius is $r = 5$ m. What is the rate of change of the area of the circle at that moment?

Recall: area of a circle

$$= A = \pi \cdot r^2 \qquad / \frac{d}{dt}[\ldots]$$

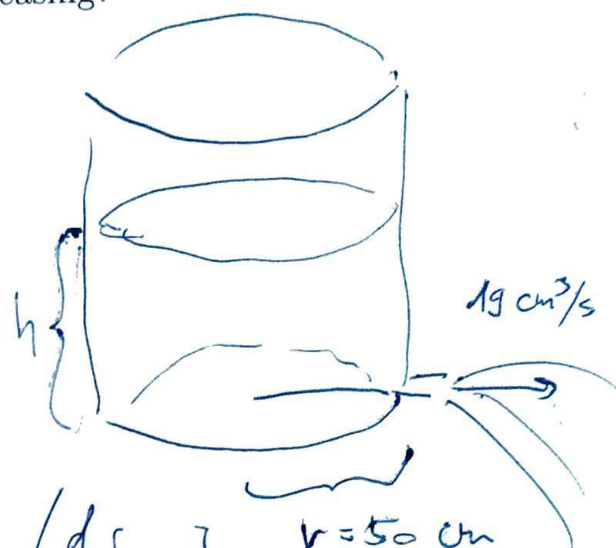


$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

Known: $r = 5$, $\frac{dr}{dt} = 3$

$$\rightsquigarrow \frac{dA}{dt} = 2 \cdot \pi \cdot 5 \cdot 3 = 30\pi \approx 94.25 \ m^2/s$$

Exercise: A water tank has a shape of a cylinder, with radius of the base 50 cm. Water escapes through a hole at the bottom of the tank at the rate 19 cm^3/s. At what rate is the water level decreasing?



Volume of water in the tank:

$$V = (\pi \cdot r^2)\cdot h \underset{\substack{\uparrow \\ r=50}}{=} 2500\cdot\pi\cdot h \quad \Big/ \frac{d}{dt}[\dots]$$

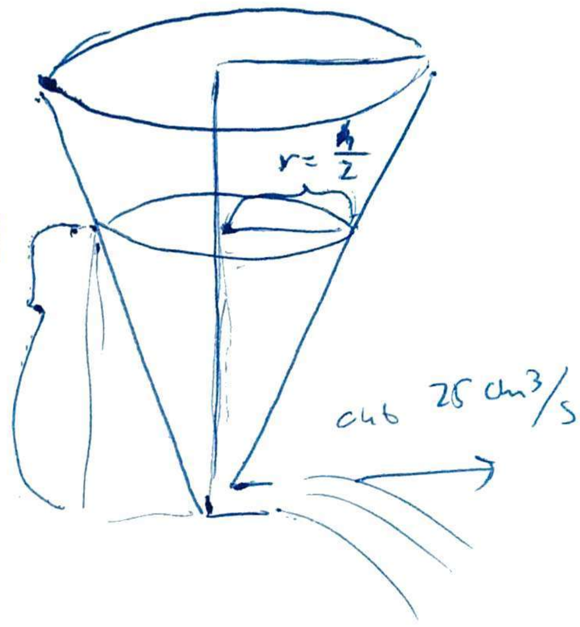
$$\frac{dV}{dt} = 2500\cdot\pi\cdot\frac{dh}{dt}$$

$\leadsto$ $(-19)\ \mathrm{cm}^3/\mathrm{s}$

$$-19 = 2500\pi\,\frac{dh}{dt}$$

$$\leadsto \frac{dh}{dt} = -\frac{19}{2500\pi}\ \mathrm{cm/s} \approx -0.00242\ \mathrm{cm/s}$$

Exercise: A water tank has a shape of a cone (pointing down), and the diameter fo the tank is equal to its altitude. Water escapes through a hole at the bottom of the tank at the rate 25 cm^3/s. When the water level is 125 cm, at what rate is the water level decreasing?



This time

$$\text{Volume} = V = \frac{1}{3}\pi r^2 \cdot h =$$

$$V = \frac{1}{3}\pi \cdot \left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi}{12} \cdot h^3 \quad \Big/ \frac{d}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3 \cdot h^2 \cdot \frac{dh}{dt} = \frac{\pi}{4} h^2 \cdot \frac{dh}{dt}$$

Know: water level $h = 125$ cm

$$\frac{dV}{dt} = -25\ \text{cm}^3/\text{s} \quad (\text{rate of change of volume})$$

$$\rightsquigarrow \quad -25 = \frac{\pi}{4} \cdot 125^2 \cdot \frac{dh}{dt}$$

$$\rightsquigarrow \quad \frac{dh}{dt} = \frac{-25}{125^2} \cdot \frac{4}{\pi} \approx \underline{-0.00204\ \text{cm}/\text{s}}$$