MA 16010 Lesson 14: Implicit Differentiation

have,

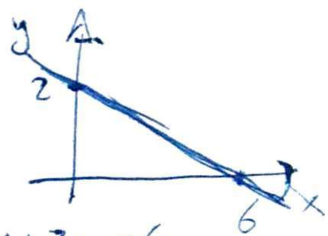
Explicit vs. Implicit functions. A function in **explicit form** is what we considered so far. It is given by an equation of the form:

$$y = f(x)$$

A function in **implicit form** is given by a more general equation involving x and y. We still think of $y = y(x)$ as a function of x.

Example: The function $y(x)$ given by the equation

$$x + 3y = 6$$



is in implicit form. The explicit form of the function is:

$$y = 2 - \frac{1}{3}x$$

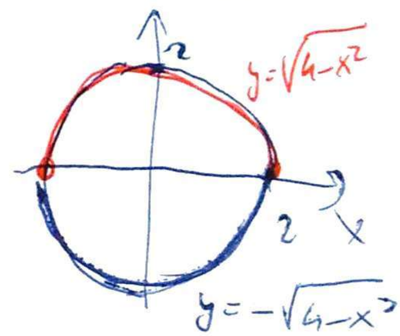
$$x + 3y = 6$$
$$3y = 6 - x$$
$$y = \frac{1}{3}(6 - x) =$$
$$= 2 - \frac{1}{3}x$$

Example: The function $y(x)$ given by the equation

$$x^2 + y^2 = 4 \quad (\text{or } \sqrt{x^2 + y^2} = 2)$$

is in implicit form. The explicit form of the function is:

$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2$$
$$y = \pm\sqrt{4 - x^2}$$



either $\boxed{y = \sqrt{4 - x^2}}$, or $\boxed{y = -\sqrt{4 - x^2}}$ (two functions!)

Implicit differentiation. Sometimes it is not easy/possible to find explicit form out of implicit one, but we can still take the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$.

Idea: Differentiate both sides of the equation with respect to x. Treat $y = y(x)$ as a function of x, and use the chain rule wherever appropriate, i.e.

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[h(y)\right] = h'(y) \cdot y'}$$

$$\frac{d}{dx} h(y(x))$$

In the end, solve for y'.

Ex 0: $x^2 + y^2 = 4 \;/\; \frac{d}{dx}[\ldots]$ $\Rightarrow$ $2yy' = -2x$

$2x + 2y \cdot y' = 0$

$\underline{y'} = -\frac{2x}{2y} = \underline{\underline{-\frac{x}{y}}}$

y'

Exercise: Using implicit differentiation, find $\frac{dy}{dx}$ when

$$4x^3 + 2xy^2 = 3y^3 - 7yx^2 \; .$$

$/ \frac{d}{dx}[\ldots]$

faktor factor

$12x^2 + (2y^2 + 2x(2y)\cdot y') = 9y^2 \cdot y' - (7y' \cdot x^2 + 7y \cdot 2x)$

$12x^2 + 2y^2 + 4xy \cdot y' = 9y^2 \cdot y' - 7x^2 y' - 14xy$

$4xy \cdot y' - 9y^2 \cdot y' + 7x^2 y' = -12x^2 - 2y^2 - 14xy$

$(4xy - 9y^2 + 7x^2) y' = -12x^2 - 2y^2 - 14xy$

$y' = \frac{-12x^2 - 2y^2 - 14xy}{4xy - 9y^2 + 7x^2} = -\frac{12x^2 + 2y^2 + 14xy}{4xy - 9y^2 + 7x^2}$

Exercise: Using implicit differentiation, find $\frac{dy}{dx}$ when

$$\cos(3x + 2y) = 5x^2 y \; .$$

$/ \frac{d}{dx}[\ldots]$

$-\sin(3x+2y)\cdot(3 + 2y') = 10x \cdot y + 5x^2 \cdot y'$

$-3\sin(3x+2y) - 2\sin(3x+2y) \cdot y' = 10xy + 5x^2 \cdot y'$

$-2\sin(3x+2y) \cdot y' - 5x^2 \cdot y' = 3\sin(3x+2y) + 10xy$

$(-2\sin(3x+2y) - 5x^2) y' = 3\sin(3x+2y) + 10xy$

$y' = \frac{3\sin(3x+2y) + 10xy}{-2\sin(3x+2y) - 5x^2} = -\frac{3\sin(3x+2y) + 10xy}{2\sin(3x+2y) - 5x^2}$

$$\csc^2\left(\frac{x}{y}\right) = \frac{1}{\sin^2\left(\frac{x}{y}\right)}$$

Exercise: Using implicit differentiation, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$3\cot\left(\frac{x}{y}\right) = 5x \quad . \qquad \Big/ \frac{d}{dx}\left[\ldots\right]$$

$$3\left(-\csc^2\left(\frac{x}{y}\right)\right)\cdot\frac{d}{dx}\left[\frac{x}{y}\right] = 5$$

$$-3\csc^2\left(\frac{x}{y}\right)\cdot\frac{1\cdot y - y'\cdot x}{y^2} = 5$$

$$-3\csc^2\left(\frac{x}{y}\right)\left(\frac{1}{y} - \frac{x}{y^2}\cdot y'\right) = 5$$

$$-\frac{3\csc^2\left(\frac{x}{y}\right)}{y} + \frac{3x}{y^2}\cdot\csc^2\left(\frac{x}{y}\right)y' = 5$$

$$\frac{3x}{y^2}\cdot\csc^2\left(\frac{x}{y}\right)y' = 5 + \frac{3\csc^2\left(\frac{x}{y}\right)}{y}$$

$$\csc^2\left(\frac{x}{y}\right)y' = \frac{5y^2}{3x} + \frac{\cancel{3}y^{\cancel{2}}\csc^2\left(\frac{x}{y}\right)}{\cancel{3}x\cdot\cancel{y}}$$

$$\csc^2\left(\frac{x}{y}\right)\cdot y' = \frac{5y^2}{3x} + \frac{y}{x}\cdot\csc^2\left(\frac{x}{y}\right)$$

$$y' = \frac{5y^2}{3x}\cdot\sin^2\left(\frac{x}{y}\right) + \frac{y}{x}$$

Exercise: Find the slope of the tangent line to $3x^2 + 2y^2 = 14$ at $(2, 1)$.

slope of the tangent line $= y'$ at $x = 2$ $(y = 1)$

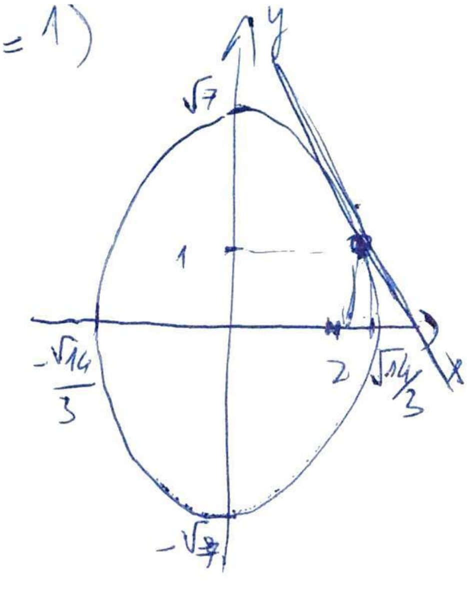
$$3x^2 + 2y^2 = 14 \quad \Big/ \frac{d}{dx}\left[\ldots\right]$$

$$6x + 4\cdot y\cdot y' = 0$$

$$4y\,y' = -6x$$

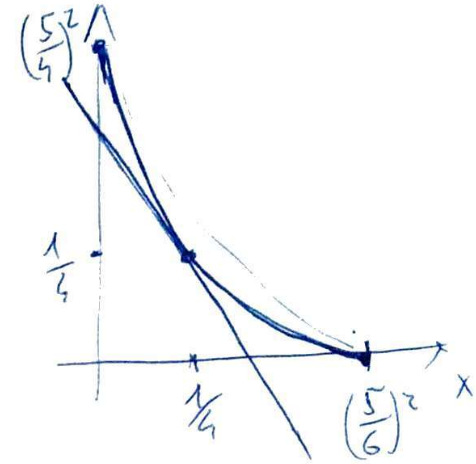
$$y' = -\frac{6x}{4y} = -\frac{3x}{2y}$$

$$y' = -\frac{3\cdot 2}{2\cdot 1} = -3$$



Exercise: Find the equation of the tangent line to $6\sqrt{x}+4\sqrt{y}=5$ at $(x,y)=(1/4,1/4)$.

1. slope of the line:



$$6\sqrt{x}+4\sqrt{y}=5 \quad \Big|\ \frac{d}{dx}[\ldots]$$

$$6x^{\frac{1}{2}}+4y^{\frac{1}{2}}=5 \quad \Big|\ \frac{d}{dx}[\ldots]$$

$$6\cdot\frac{1}{2}\cdot x^{-\frac{1}{2}}+4\cdot\frac{1}{2}\cdot y^{-\frac{1}{2}}\cdot y'=0$$

$$\frac{3}{\sqrt{x}}+\frac{2}{\sqrt{y}}\cdot y'=0$$

$$\frac{2}{\sqrt{y}}\cdot y'=-\frac{3}{\sqrt{x}}$$

$$y'=-\frac{3}{\sqrt{x}}\cdot\frac{\sqrt{y}}{2}=-\frac{3}{2}\cdot\frac{\sqrt{y}}{\sqrt{x}}, \quad x=\frac{1}{4},\ y=\frac{1}{4}$$

$$y'=-\frac{3}{2}\cdot\frac{\sqrt{1/4}}{\sqrt{1/4}}=-\frac{3}{2}$$

2. line of the form $y=-\frac{3}{2}x+b$, passes through $\left(x=\frac{1}{4}, y=\frac{1}{4}\right)$

$\rightsquigarrow$ $\frac{1}{4}=-\frac{3}{2}\left(\frac{1}{4}\right)+b$

$b=\frac{1}{4}+\frac{3}{2}\cdot\frac{1}{4}=2\cdot\frac{1}{4}=\frac{1}{2}$

$\rightsquigarrow$ $y=-\frac{3}{2}x+\frac{1}{2}$

OR

2. „point-slope formula"

$y-y_0=(\text{slope})\cdot(x-x_0)$

$\rightsquigarrow$ $y-\frac{1}{4}=\left(-\frac{3}{2}\right)\left(x-\frac{1}{4}\right)$

$y=-\frac{3}{2}\left(x-\frac{1}{4}\right)+\frac{1}{4}$

$y=-\frac{3}{2}x+\frac{3}{2}\cdot\frac{1}{4}+\frac{1}{4}$

$y=-\frac{3}{2}x+\frac{1}{2}$