MA 16010 Lesson 12: Chain rule II & logarithm

Recall (chain rule): Given two functions $y = f(u)$ and $u = g(x)$, for their composition $y = f(g(x))$ we have

$$y'(x) = f'(g(x)) \cdot g'(x) \quad , \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad .$$

Today we compute some more complicated derivatives, using all our rules.

Exercise: Compute the derivative of $h(x) = (3x-4)^4(x^3-5)^2$.

$$h'(x) = \underbrace{\frac{d}{dx}\left[(3x-4)^4\right]} \cdot (x^3-5)^2 + (3x-4)^4 \cdot \underbrace{\frac{d}{dx}\left[(x^3-5)^2\right]} \quad \text{(product rule)}$$

chain rule: $f(u) = u^4$, $f'(u) = 4u^3$, $g(x) = 3x-4$, $g'(x) = 3$

chain rule: $f(u) = u^2$, $f'(u) = 2u$, $g(x) = x^3-5$, $g'(x) = 3x^2$

$$= 4\cdot(3x-4)^3 \cdot 3 \cdot (x^3-5)^2 + (3x-4)^4 \cdot 2(x^3-5) \cdot 3x^2 =$$

$$= 6(3x-4)^3(x^3-5) \cdot (12x^3 - 40 + (3x-4)\cdot x^2) =$$

$$= 6(3x-4)^3(x^3-5)(2x^3 - 10 + 3x^3 - 4x^2) = \underline{\underline{6(3x-4)^3(x^3-5)(5x^3-4x^2-10)}}$$

Exercise: Compute the derivative of $h(x) = \frac{\sqrt{16-x^2}}{3x}$ at $x = 1$.

$$h'(x) = \frac{\frac{d}{dx}\left(\sqrt{16-x^2}\right)\cdot 3x - 3\cdot\sqrt{16-x^2}}{(3x)^2}$$

$\frac{d}{dx}\left(\sqrt{16-x^2}\right)$ $\quad f(u) = \sqrt{u} = u^{\frac{1}{2}}$ $\rightsquigarrow f'(u) = \frac{1}{2}u^{-\frac{1}{2}}$

$g(x) = 16 - x^2 \rightsquigarrow g'(x) = -2x$

$$h'(x) = \frac{\frac{1}{2}(16-x^2)^{-\frac{1}{2}}\cdot(-2x)\cdot 3x - 3\sqrt{16-x^2}}{(3x)^2}$$

$$h'(x) = \frac{-3x^2\cdot(16-x^2)^{-\frac{1}{2}} - 3\sqrt{16-x^2}}{9x^2} = -\frac{1}{3\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{3x^2}$$

$$h'(1) = -\frac{1}{3\sqrt{15}} - \frac{\sqrt{15}}{3} \approx -1.377$$

Exercise: Compute the derivative of $h(x) = e^{5x}\cot(7x)$.

$$h'(x) = \frac{d}{dx}\left[e^{5x}\right]\cdot\cot(7x) + e^{5x}\frac{d}{dx}\left[\cot(7x)\right] =$$

$$= e^{5x}\cdot 5\cdot\cot(7x) + e^{5x}\cdot\left(-\csc^2(7x)\right)\cdot 7$$

$$= \underline{\underline{e^{5x}}}\left(5\cdot\cot(7x) - 7\csc^2(7x)\right)$$

Derivative of the natural logarithm. We can use the chain rule to figure out what is the derivate of $y = \ln(x)$:

$$e^{\ln(x)} = x \qquad / \frac{d}{dx}[\dots] \text{ on both sides}$$

$$\underbrace{e^{\ln(x)}}_{=x}\cdot\frac{d}{dx}\left[\ln(x)\right] = 1$$

Chain rule:

$f(u) = e^u \leadsto f'(u) = e^u$

$g(x) = \ln(x) \leadsto g'(x) = ?$

(this is what we need)

$$x\cdot\frac{d}{dx}\left[\ln(x)\right] = 1$$

$$\underline{\underline{\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}}}$$

Summary: $\boxed{\dfrac{\mathrm{d}}{\mathrm{d}x}\Big[\ln(x)\Big] = \dfrac{1}{x}.}$

chain rule:

$f(u) = \ln(u) \rightsquigarrow f'(u) = \frac{1}{u}$

$g(x) = x^3 + 3x \rightsquigarrow g'(x) = 3x^2 + 3$

Exercise: Compute $y'(2)$ when $y = \ln(x^3 + 3x)$.

$$y'(x) = \frac{1}{x^3+3x} \cdot (3x^2+3) = \underline{\frac{3x^2+3}{x^3+3x}}$$

$$y'(2) = \frac{3\cdot 2^2+3}{2^3+3\cdot 2} = \frac{15}{14}$$

Exercise: Compute $y'(x)$ when $y = \cos(2x^2 + 5)\ln(33x)$.

$$y'(x) = \underbrace{\frac{d}{dx}\left[\cos(2x^2+5)\right]} \cdot \ln(33x) + \cos(2x^2+5) \cdot \underbrace{\frac{d}{dx}\left[\ln(33x)\right]}$$

$f(u) = \cos(u) \rightsquigarrow f'(u) = -\sin u$

$g(x) = 2x^2+5 \rightsquigarrow g'(x) = 4x$

$f(u) = \ln(u) \rightsquigarrow f'(u) = \frac{1}{u}$

$g(x) = 33x \rightsquigarrow g'(x) = 33$

$$= -\sin(2x^2+5)\cdot 4x \cdot \ln(33x) + \cos(2x^2+5)\cdot \frac{1}{\cancel{33}x}\cdot \cancel{33}$$

$$= -4\sin(2x^2+5)\cdot x \cdot \ln(33x) + \frac{\cos(2x^2+5)}{x}$$

Exercise: Compute $h'(x)$ when $h(x) = 5^x$.

$$h(x) = 5^x = \left(e^{\ln(5)}\right)^x = e^{\ln(5)\cdot x}$$

$\frac{d}{dx}[e^x] = e^x$

! Not true for $y = a^x$ where $a \neq e$

$$h'(x) = \underbrace{e^{\ln(5)\cdot x}}_{5^x} \cdot \ln(5) = \underline{5^x \cdot \ln(5)}$$

Chain rule:

$f(u) = e^u \rightsquigarrow f'(u) = e^u$

$g(x) = \ln(5)\cdot x \rightsquigarrow g'(x) = \ln(5)$

(in general:

$$\frac{d}{dx}\left[a^x\right] = a^x \cdot \ln(a)$$)

Exercise: The position of a particle moving on a straight line is given by

$$s(t) = \frac{50t}{(30 + 2t^2)^2}$$

(in meters, where t is time in seconds). What is the velocity of the particle at $t = 4$ seconds? Round your answer to 3 decimal places.

$$v(t) = \frac{ds}{dt} = \frac{50 \cdot (30+2t^2)^2 - 50t \cdot \frac{d}{dt}\left[(30+2t^2)^2\right]}{(30+2t^2)^4} \quad \text{(quotient rule)}$$

Chain rule:

$f(u) = u^2 \rightarrow f'(u) = 2u$

$g(t) = 30+2t^2$

$\rightarrow g'(t) = 4t$

$$v(t) = \frac{50(30+2t^2)^2 - 50t \cdot 2 \cdot (30+2t^2) \cdot 4t}{(30+2t^2)^4}$$

$$v(t) = \frac{50(30+2t^2) - 400t^2}{(30+2t^2)^3}$$

$$v(3) = \frac{50 \cdot (30+18) - 400 \cdot 9}{(30+18)^3} \approx -0.011 \text{ m/s}$$