MA 16010 Lesson 11: Chain rule I

Recall (composition of functions): Given two functions $f(u)$ and $g(x)$, their composition is the function $y = \underbrace{f(\underbrace{g(x)}_{\text{inner function}})}_{\text{outer function}}$.

Question for today: How to compute the derivative of a composite function in terms of the original functions?

Example: Compute the derivative of $h(x) = (x + \sin(x))^3$.

We have $h(x) = f(g(x))$ where $f(u) = u^3$ (so $f'(u) = 3u^2$),

and $g(x) = x + \sin(x)$ (so $g'(x) = 1 + \cos(x)$).

Using product rule (slow, complicated):

$$h'(x) = \frac{d}{dx}\left[(x+\sin(x))\cdot(x+\sin x)^2\right] = (1+\cos(x))(x+\sin x)^2 + (x+\sin(x))\frac{d}{dx}\left[(x+\sin x)^2\right]$$

$$= (1+\cos(x))(x+\sin x)^2 + (x+\sin x)\left((1+\cos x)(x+\sin x) + (x+\sin x)(1+\cos x)\right)$$

$$= (1+\cos(x))(x+\sin x)^2 + (x+\sin x)(1+\cos x)(x+\sin x) + (x+\sin x)^2(1+\cos x)$$

$$= \underbrace{3\cdot(x+\sin x)^2}_{f'(g(x))}\cdot\underbrace{(1+\cos x)}_{g'(x)}$$

Chain rule: $\boxed{\dfrac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))\cdot g'(x)}$

Exercise: Compute $y'(x)$ when $y = (x^{100} + 4)^{1000}$.

$y = f(g(x))$ for $f(u) = u^{1000}$ (outer function) $\rightsquigarrow f'(u) = 1000u^{999}$

$g(x) = x^{100} + 4$ (inner function) $\rightsquigarrow g'(x) = 100x^{99}$

By the chain rule,

$$y'(x) = f'(g(x))\cdot g'(x) = 1000\cdot(x^{100}+4)^{999}\cdot 100\cdot x^{99} = \underline{\underline{100\,000\cdot(x^{100}+4)^{999}\cdot x^{99}}}$$

Another way to remember the chain rule:
Consider functions $y = f(u)$ and $u = g(x)$. We may consider $y = f(g(x))$ to be the composite function. Then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

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Exercise: Use the chain rule to compute $h'(x)$ when:

$h(x) = (\cos(x) + \tan(x))^{-5}$: $f(u) = u^{-5} \rightsquigarrow f'(u) = -5u^{-6}$

$g(x) = \cos(x) + \tan x \rightsquigarrow g'(x) = -\sin x + \sec^2(x)$

$$h'(x) = -5 \cdot (\cos(x) + \tan(x))^{-6} \cdot (-\sin(x) + \sec^2(x)) =$$

$$= 5 \cdot (\cos(x) + \tan x)^{-6} \cdot (\sin(x) - \sec^2(x))$$

$h(x) = \sqrt[3]{(x^7 + 8)}$: $f(u) = \sqrt[3]{u} = u^{\frac{1}{3}} \rightsquigarrow f'(u) = \frac{1}{3} \cdot u^{-2/3}$

$g(x) = x^7 + 8 \rightsquigarrow g'(x) = 7x^6$

$$h'(x) = \frac{1}{3} \cdot (x^7 + 8)^{-\frac{2}{3}} \cdot 7x^6 = \frac{7}{3}(x^7 + 8)^{-\frac{2}{3}} \cdot x^6$$

$h(x) = \left(\frac{3x}{x+5}\right)^8$: $f(u) = u^8 \rightsquigarrow f'(u) = 8u^7$

$$g(x) = \frac{3x}{x+5} \rightsquigarrow g'(x) = \frac{3 \cdot (x+5) - 1 \cdot 3x}{(x+5)^2} =$$

$$= \frac{3x + 15 - 3x}{(x+5)^2} = \frac{15}{(x+5)^2}$$

$$h'(x) = 8 \cdot \left(\frac{3x}{x+5}\right)^7 \cdot \frac{15}{(x+5)^2} = 120 \cdot \frac{3^7 \cdot x^7}{(x+5)^7 (x+5)^2} = 262440 \frac{x^7}{(x+5)^9}$$

$$\frac{(3x)^7}{(x+5)^7} = \frac{3^7 \cdot x^7}{(x+5)^7}$$

Exercise: Compute $h'(\ln(\pi))$ for $h'(x)$ at $x = \ln(\pi)$

$$h(x) = \cos(e^x + \pi/2).$$

$f(u) = \cos(u) \rightsquigarrow f'(u) = -\sin(u)$

$g(x) = e^x + \frac{\pi}{2} \rightsquigarrow g'(x) = e^x$

$h'(x) = -\sin\left(e^x + \frac{\pi}{2}\right) \cdot e^x$

$h'(\ln(\pi)) = -\sin\left(\underbrace{e^{\ln(\pi)}}_{\pi} + \frac{\pi}{2}\right) \cdot \underbrace{e^{\ln(\pi)}}_{\pi} =$

$= -\sin\left(\frac{3}{2}\pi\right) \cdot \pi$

$= -(-1) \cdot \pi =$

$= \underline{\underline{\pi}}$

Exercise: Compute the derivative $h'(x)$ for

$$h(x) = e^{200x} = \left(e^x\right)^{200}$$

using the chain rule in two different ways.

1st way: $f(u) = e^u \rightsquigarrow f'(u) = e^u$

$g(x) = 200x \rightsquigarrow g'(x) = 200$

$h'(x) = e^{200x} \cdot 200 = \underline{\underline{200e^{200x}}}$

2nd way: $f(u) = u^{200} \rightsquigarrow f'(u) = 200u^{199}$

$g(x) = e^x \rightsquigarrow g'(x) = e^x$

$h'(x) = 200\left(e^x\right)^{199} \cdot e^x = 200\left(e^x\right)^{199+1} = 200\left(e^x\right)^{200}$

$= \underline{\underline{200e^{200x}}}$