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Spectral Properties of Families of Matrices

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Introduction

Central topic of the thesis is the study of spectral properties of sets of matrices. We focus our attention on three classes of families of matrices:

- families generated by rank–one perturbations of a fixed complex matrix
- families of stochastic matrices
- families of generic square matrices

About the first class we are particulary interested in analysing the spectral properties of matrices perturbed using one of their eigenvectors, namely, given a generic complex square matrix A and nonzero complex vectors x and v such that $Ax = \lambda x$ and $v^*x = 1$, we want to study the spectral properties of $A(c) = cA + (1-c)\lambda xv^*$ as functions of the complex variable c. This class of perturbed matrices includes a distinguished one: the Google matrix, i.e. the matrix associated with the Google PageRank model.

Since the publication of the paper "The Page–Rank citation ranking: bringing order to the web" in 1998 [37] and the foundation of the Google enterprise by Sergey Brin and Larry Page the PageRank model has become a essential instrument for an effective web search. Before the introduction of this algorithm all web search engines, like AltaVista, Yahoo! or Northern Light, they were unable to sort the results of a specific web search in a convenient way: it is almost useless to obtain, as a result of a web search, one thousand of web pages randomly ordered, no one will look for a good answer to his/her request among all these results. Nowadays it is normal to find more than fifty thousands web pages related to a specific web search, this is due to the fact that the number of pages, which are currently present on the internet, is estimated around ten billions and is consistently increasing.

As a matter of fact this model could be of interest not only in Web ranking, but also for ranking human resources as well as in many aspects of marketing, in political/social sciences e.g. for ranking who/what is influential and who/what is not, for ranking the importance of a paper and/or of a researcher looking in scientific databases (see [4]) etc.

However, even if the PageRank works undoubtedly well, there are pathological behaviors and limitations of the actual model, pointed out in this thesis, that can reduce considerably the adherence to the reality of the results. For this reason we propose possible improvements to the model that remove the old pathologies without introducing new ones and preserving its efficiency (fast computation).

Furthermore we make use of the theoretic results concerning rank–one perturbed matrices, presented in the first Chapter, to shed new light on the eigenvalues and eigenvector structure of the original Google matrix $G(c) = cG + (1-c)\lambda x v^T$ as functions of the parameter c (with G the *basic Google matrix*, $\lambda = 1$, $x = [1 \ 1 \ \cdots \ 1]^T$, and v a nonnegative probability vector). In particular we would like to understand the behavior (regularity, expansions, limits, conditioning etc.) of the left 1–eigenvector $y(c)$ of $G(c)$, the so–called PageRank, as a function of c especially in the limit case of c close or equal to 1.

Our interest in computing the PageRank $y(c)$ with c as close as possible to 1 is related to the role of this parameter: the more c is close to 1 the more the computed PageRank, besides being meaningful in terms of surfing model, is meaningful in terms of the notion of importance (in the limit $y(1)$ is exactly a generalization of the notion, common in social sciences and daily life, of very important person, as clarified in Chapter 2). Therefore the evaluation of the PageRank $y(1)$ gains in importance from the perspective of using this kind of models not only in search engines optimization, but also in other contexts.

It is known that for $c = 1$ the problem is ill–posed since there exist infinitely many left eigenvectors $y(1)$, with l^1 norm equal 1, which form a convex set. On the other hand, for $c \in [0,1)$, the solution exists and is unique, but the known algorithms become very slow when c is close to 1. Nevertheless we want to compute $y(c)$ in these difficult cases, especially in the limit as c tends to 1; to do so we make use of the analytical characterization of $y(c)$, which we obtain from the study of the $G(c)$ spectral properties, and what we discover and present in the following is, in our opinion, extremely interesting and promising.

About the second class we are interested in analysing products of stochastic matrices which are associated with a model, proposed by Vicsek et al. in 1995 [50], for the coordination of groups of autonomous agents. This model allows to represent the behavior of a wide range of biological swarming systems like herds of quadrupeds, schools of fish, flocks of flying birds, bacterial colonies [50] and also the behavior of artificial systems like groups of unmanned aerial vehicles, mobile robots or networks of sensors [103]. More in general it allows to solve distributed agreement problems: given a system of agents this model provides a protocol that enables the agents to agree upon quantities of interest via a process of distributed decision making. Our aim is to study sufficient conditions that ensure the convergence to a global consensus among the autonomous agents. This is made possible, as suggested by Jadbabaie et al. in [89], studying the spectral properties of the family of stochastic matrices associated with the system. The model proves to be related to the PageRank one and its analysis is based on algorithms, developed for generic families of matrices, which we present in the third part of the thesis.

Regarding the last case, during the past few decades there has been an increasing interest in studying the behavior of long products generated using matrices of a given generic family and in particular in analysing the maximal growth rate of these products. This study can be done considering the generalization of the spectral radius of a matrix to the case of a family of matrices, which is called *joint spectral radius* or simply *spectral radius* and it was first introduced by Rota and Strang in the three pages paper "*A note on the joint spectral radius*" published in 1960 [110]. To be precise this generalization can be formulated in many different ways, but for the families of matrices which are common in applications, i.e. bounded and finite, all the possible generalizations coincide with each other in a unique value that is called joint spectral radius, as explained in Chapter 4.

The joint spectral radius analysis proves to be useful in many different contexts like, for example, in the construction of wavelets of compact support [70], in analysing the asymptotic behavior of solutions of linear difference equations with variable coefficients [77, 78, 79], in the coordination of autonomous agents [89, 103, 73] and many others [113].

The same quantity, however, can prove to be hard to compute and can lead even to undecidable problems [63, 54]. In this thesis we present all the known generalizations of spectral radius, the properties, theoretical results and challenges associated with them and an algorithm for the exact evaluation of the joint spectral radius. We make use of this algorithm to prove a finiteness conjecture about $2{\times}2$ sign–matrices proposed recently by Blondel, Jungers and Protasov [56, 91].

Thesis Outline

The work, as previously mentioned, is divided into three parts:

The first one concerns families of matrices generated by rank–one perturbations of a fixed square complex matrix. In Chapter 1 we introduce the *complete principle of biorthogonality* as a generalization of the classical Brauer's principle of biorthogonality and we use it to deduce eigenvalues and Jordan blocks of rank–one perturbed matrix $A(c) = cA + (1-c)\lambda xv^*$ with A square complex matrix, c complex variable, x and v nonzero complex vectors such that $Ax = \lambda x$ and $v^*x = 1$. We derive also a representation for a distinguished left λ–eigenvector $y(c)$ of $A(c)$ and we study the limit of $y(c)$ as $c \to 1$ in the complex plane. Chapter 2 concerns, first of all, a discussion on the Google model and on its adherence to the reality: a basic example presented in Section 2.2 it is used to point out pathologies and limitations of the actual model and to propose some possible improvements. Furthermore, given the basic (stochastic) Google matrix G, a real parameter c such that $0 < c < 1$, a nonnegative probability vector v and the all–ones vector e, it is known that for the real valued matrix $G(c) = cG + (1-c)ev^T$, there is a unique nonnegative vector $y(c)$ such that $y(c)^T G(c) = y(c)^T$ and $y(c)^T e = 1$, which is the *PageRank* vector $y(c)$; in this Chapter a complex analog of PageRank $y(c)$ is presented for the Web hyperlink matrix $G(c) = cG + (1-c)ev^*$, obtained as a special case of the results of the previous Chapter when c is a complex number and v is a complex vector such that $v^*e = 1$. We study regularity, limits, expansions, and conditioning of $y(c)$ in the complex case and we propose algorithms (e.g. complex extrapolation, power method on a modified matrix etc.) that may provide an efficient way to compute PageRank also with c close or equal to 1. An interpretation of this limit value of $y(c)$ and a critical discussion on the original model of PageRank, on its adherence to reality and on possible ways for its improvement represent our contribution on modeling issues.

The second part, Chapter 3, is devoted to the study of spectral properties of stochastic matrices with particular emphasis to the applications to the so–called consensus problem. We present the Vicsek model [50], which allows to represent the distributed coordination of autonomous agents, and the analysis of the convergence to a global consensus in the agents network. This analysis, based on the spectral properties of stochastic matrices products associated with the evolution of the system, is obtained applying techniques developed in the following Chapter. We give sufficient conditions that ensure a global consensus among the agents and provide an estimate for the maximal rate of convergence to this solution. A few simulations are presented to give the reader a foretaste of the potentiality of this approach. We present, furthermore, alternative proofs of a few Theorems on this subject presented in the literature and the connections between the Google and Vicsek model.

Third part is about generic families. In Chapter 4 the generalizations of spectral radius to the case of a set of matrices are presented. We first introduce a case of study that requires the analysis of products of matrices taken from a given set, we then present the definitions of joint, generalized, common and mutual spectral radii and we describe results and properties, related to these quantities, valid for bounded or finite sets. In particular we present the irreducibility, nondefectivity and finiteness properties. The irreducibility and, more in general, the nondefectivity of a family lead to the existence of an extremal norm for the family itself. Chapter 5 deals with the spectral radius exact computation in the case of families of matrices. We present an algorithm, based on the construction of unit balls of extremal polytope norms, that proves to be efficient in many cases and allows us to prove the finiteness conjecture for 2×2 sign–matrices.

The thesis concludes with an Appendix containing a detailed analysis of all the 2×2 sign–matrices families presented in Chapter 5 and another one concerning numerical results on the coordination of autonomous agents.

The original results presented in this thesis have been published in:

- A. Cicone, S. Serra–Capizzano. Google PageRanking problem: the model and the analysis, *Journal of Computational and Applied Mathematics* 234 (11), 2010, pp 3140–3169.

- A. Cicone, N. Guglielmi, S. Serra–Capizzano, M. Zennaro. Finiteness property of pairs of 2×2 sign–matrices via real extremal polytope norms, *Linear Algebra and its Applications* vol. 432 (2–3), 2010, pp 796–816.

Part I

Rank-one-perturbation families

Chapter 1

Theory

We consider a square complex matrix A and nonzero complex vectors x and v such that $Ax = \lambda x$ and $v^*x = 1$. We use standard matrix analytic tools to determine the eigenvalues, the Jordan blocks, and a distinguished left λ–eigenvector of $A(c) = cA + (1-c)\lambda xv^*$ as a function of a complex variable c. If λ is a semisimple eigenvalue of A, there is a uniquely determined projection YX^* such that $\lim\limits_{c\to 1} y(c) = YX^*v$ for all v; this limit may fail to exist for some v if λ is not semisimple.

In Section 1.1 we set notation and terminology for the basic matrix–theoretic concepts that we employ to analyze a parametric matrices $A(c)$: for a square complex matrix A, nonzero complex vectors x and v such that $Ax = \lambda x$ and $v^*x = 1$, and a complex variable c, we study $A(c) = cA + (1-c)\lambda xv^*$. In Section 1.2 we explain how Alfred Brauer used the *classical principle of biorthogonality* in 1952 to prove a theorem that reveals the eigenvalues of $A(c)$. In Section 1.3 we introduce the *complete principle of biorthogonality* and use it to obtain the Jordan blocks of $A(c)$ under the assumption that there is a nonzero vector y such that $y^*A = \lambda y^*$ and $y^*x = 1$. In particular, such a vector y exists if λ is a simple or semisimple eigenvalue of A. In Section 1.4 we derive a representation for a distinguished left λ–eigenvector $y(c)$ of $A(c)$; this representation is an explicit rational vector–valued function of the complex variable c. In Subsection 1.4.1 we study the behavior of the left eigenvector $y(c)$ as $c \to 1$ in the complex plane.

1.1 Terminology and notation

All the matrices and vectors that we consider have real or complex entries. We denote the *conjugate transpose* of an m–by–n matrix $X = [x_{ij}]$ by $X^* = [\bar{x}_{ji}]$, while the simple *transpose* as $X^T = [x_{ji}]$. For $p \in [1,\infty)$, the l^p norm of a vector $w \in \mathbb{C}^n$ is given by

$$\|w\|_p = \sqrt[p]{\sum_{i=1}^{n} |w[i]|^p}$$

In particular:

l^1 – The sum norm

$$\|w\|_1 = \sum_i |w[i]|$$

l^2 – The Euclidean norm

$$\|w\|_2 = \sqrt{\sum_{i=1}^{n} |w[i]|^2} = \sqrt{w^* w}$$

while its l^∞ norm, known also as *max norm*, is

$$\|w\|_\infty = \max_{j=1,\ldots,n} |w[j]|.$$

For a square matrix $A \in \mathbb{C}^{n\times n}$ and for $p \in [1,\infty]$, $\|A\|_p$ is the associated induced norm given by

$$\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p$$

If A is a square matrix, its *characteristic polynomial* is $p_A(t) := \det(tI - A)$, where det stands for determinant [24, Section 0.3]; the (complex) zeroes of $p_A(t)$ are the *eigenvalues* of A. A complex number λ is an eigenvalue of A if and only if there are nonzero vectors x and y such that $Ax = \lambda x$ and $y^*A = \lambda y^*$; x is said to be an *eigenvector* (more specifically, a *right eigenvector*) of A associated with λ and y is said to be a *left eigenvector* of A associated with λ. The set of all the eigenvalues of A is called the *spectrum* of A and is denoted by $\sigma(A)$. The *determinant* of A, $\det A$, is equivalent to the product of all its eigenvalues. If the spectrum of A does not contain 0 the matrix is said *nonsingular* (A nonsingular if and only if $\det A \neq 0$). The *spectral radius* of A is the nonnegative real number $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. If $\lambda \in \sigma(A)$, its *algebraic multiplicity* is its multiplicity as a zero of $p_A(t)$; its *geometric multiplicity* is the maximum number of linearly independent eigenvectors associated with it. The geometric multiplicity of an eigenvalue is never greater than its algebraic multiplicity. An eigenvalue whose algebraic multiplicity is one is said to be *simple*. An eigenvalue λ of A is said to be *semisimple* if and only if rank$(A-\lambda I)$ =rank$(A-\lambda I)^2$ i.e. λ has the same geometric and algebraic multiplicity. If the geometric multiplicity and the algebraic multiplicity are equal for every eigenvalue, A is said to be *nondefective*, otherwise is *defective*.

We let e_1 indicate the first column of the identity matrix I: $e_1 = [1\ 0\ \cdots\ 0]^T$. We let $e = [1\ 1\ \cdots\ 1]^T$ denote the all–ones vector. Whenever it is useful to indicate that an identity or zero matrix has a specific size, e.g., r–by–r, we write I_r or 0_r.

Two vectors x and y of the same size are *orthogonal* if $x^*y = 0$. The *orthogonal complement* of a given set of vectors is the set (actually, a vector space) of all vectors that are orthogonal to every vector in the given set.

An n–by–r matrix X has *orthonormal* columns if $X^*X = I_r$. A square matrix U is *unitary* if it has orthonormal columns, that is, if U^* is the inverse of U.

A square matrix A is a *projection* if $A^2 = A$.

A square matrix A is *row–stochastic* if it has real nonnegative entries and $Ae = e$, which means that the sum of the entries in each row is 1; A is *column–stochastic* if A^T is row–stochastic. We say that A is *stochastic* if it is either row–stochastic or column–stochastic.

The *direct sum* of k given square matrices $X_1, \ldots, X_k$ is the block diagonal matrix

$$\begin{bmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{bmatrix} = X_1 \oplus \cdots \oplus X_k.$$

The k–by–k *Jordan block* with eigenvalue λ is

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}, \quad J_1(\lambda) = [\lambda].$$

Each square complex matrix A is similar to a direct sum of Jordan blocks, which is unique up to permutation of the blocks; this direct sum is the *Jordan canonical form* of A. The algebraic multiplicity of λ as an eigenvalue of $J_k(\lambda)$ is k; its geometric multiplicity is 1. If λ is a semisimple eigenvalue of A with multiplicity m, then the Jordan canonical form of A is $\lambda I_m \oplus J$, in which J is a direct sum of Jordan blocks with eigenvalues different from λ; if λ is a simple eigenvalue, then $m = 1$ and the Jordan canonical form of A is $[\lambda] \oplus J$. A is diagonalizable, i.e. its Jordan canonical form is given by a diagonal matrix, if and only if is nondefective.

Suppose that a square matrix A is similar to the direct sum of a zero matrix and a nonsingular matrix, that is,

$$A = S \begin{bmatrix} 0_m & 0 \\ 0 & B \end{bmatrix} S^{-1}, \ B \text{ is nonsingular.} \tag{1.1}$$

The matrix

$$A^D = S \begin{bmatrix} 0_m & 0 \\ 0 & B^{-1} \end{bmatrix} S^{-1}$$

is called the *Drazin inverse* of A; it does not depend on the choice of S or B in the representation (1.1). [16, Chapter 7] Moreover, both $AA^D = A^D A$ and $I - AA^D$ are projections. If X and Y have m columns, $S = [X \ S_2]$, and $(S^{-1})^* = [Y \ Z_2]$, then $A^D = S_2 B^{-1} Z_2^*$ and $I - AA^D = XY^*$.

In a block matrix, the symbol $\bigstar$ denotes a block whose entries are not required to take particular values. Finally we consider $A^0 = I$. For a systematic discussion of a broad range of matrix analysis issues, see [24].

1.2 Basic principle of biorthogonality and eigenvalues

The following observation about left and right eigenvectors associated with different eigenvalues is the *basic principle of biorthogonality* [24, Theorem 1.4.7].

1.2.1 Lemma. *Let A be a square complex matrix and let x and y be nonzero complex vectors such that $Ax = \lambda x$ and $y^*A = \mu y^*$. If $\lambda \neq \mu$, then $y^*x = 0$ (that is, x and y are orthogonal).*

Proof. Compute y^*Ax in two ways: (i) as $y^*(Ax) = y^*(\lambda x) = \lambda(y^*x)$, and (ii) as $(y^*A)x = (\mu y^*)x = \mu(y^*x)$. Since $\lambda(y^*x) = \mu(y^*x)$ and $\lambda \neq \mu$, it follows that $y^*x = 0$. □

For a given vector v and a matrix A with eigenvalue λ and associated eigenvector x, how are the eigenvalues of $A + xv^*$ related to those of A? This question was asked and answered by Alfred Brauer in 1952 [11, Theorem 26]:

1.2.2 Theorem (Brauer). *Let A be an n–by–n complex matrix and let x be a nonzero complex vector such that $Ax = \lambda x$. Let*

$$\lambda, \lambda_2, \ldots, \lambda_n$$

be the eigenvalues of A. Then for any complex n–vector v the eigenvalues of $A + xv^*$ are

$$\lambda + v^*x, \lambda_2, \ldots, \lambda_n.$$

Brauer's proof involved three steps:

(i) Compute

$$(A + xv^*)x = Ax + xv^*x = \lambda x + (v^*x)x = (\lambda + v^*x)x,$$

which shows that $\lambda + v^*x$ is an eigenvalue of $A + xv^*$.

(ii) If μ is an eigenvalue of A that is different from λ, and if y is a left eigenvector of A associated with μ, then Lemma 1.2.1 ensures that

$$y^*(A + xv^*) = y^*A + y^*xv^* = \mu y^* + (y^*x)v = \mu y^* + 0 \cdot v = \mu y^*.$$

Thus, the distinct eigenvalues of A that are different from λ are all eigenvalues of $A+xv^*$, but perhaps not with the same multiplicities.
(iii) Brauer completed his proof with a continuity argument to show that the multiplicities of the common eigenvalues of A and $A+xv^*$ (setting aside the respective eigenvalues λ and $\lambda+v^*x$) are the same.

Brauer's theorem tells us something interesting about the eigenvalues of $A(c)$.

1.2.3 Corollary. *Let A be an n–by–n complex matrix. Let λ be an eigenvalue of A, let x and v be nonzero complex vectors such that $Ax=\lambda x$ and $v^*x=1$, and let $A(c)=cA+(1-c)\lambda xv^*$. Let*

$$\lambda, \lambda_2, \ldots, \lambda_n$$

be the eigenvalues of A. Then for any complex number c, the eigenvalues of $A(c)$ are

$$\lambda, c\lambda_2, \ldots, c\lambda_n.$$

Proof. In the statement of Brauer's Theorem, replace A and v by cA and $(1-\bar{c})\bar{\lambda}v$, respectively. The eigenvalues of cA are $c\lambda, c\lambda_2, \ldots, c\lambda_n$, x is an eigenvector of cA associated with the eigenvalue $c\lambda$, and Brauer's Theorem tells us that the eigenvalues of $cA+x((1-\bar{c})\bar{\lambda}v)^* = cA+(1-c)\lambda xv^*$ are $c\lambda + (1-c)\lambda v^*x, c\lambda_2, \ldots, c\lambda_n$, which are $\lambda, c\lambda_2, \ldots, c\lambda_n$ since $v^*x=1$. □

Robert Reams [38, p. 368] revisited Brauer's theorem in 1996. He observed that the Schur triangularization theorem [24, Theorem 2.3.1] can be used to prove Brauer's Theorem without a continuity argument: Let $S=[x\ S_1]$ be any nonsingular matrix that upper triangularizes A as

$$S^{-1}AS = \begin{bmatrix} \lambda & \bigstar & \cdots & \bigstar \\ & \lambda_2 & \ddots & \vdots \\ & & \ddots & \bigstar \\ 0 & & & \lambda_n \end{bmatrix}$$

and whose first column is an eigenvector x associated with the eigenvalue λ.

Since $I = S^{-1}S = [S^{-1}x \quad \bigstar]$, we see that $S^{-1}x = e_1$. Compute

$$\begin{aligned}
S^{-1}(xv^*)S &= \left(S^{-1}x\right)(v^*S) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} v^*x & \bigstar & \cdots & \bigstar \end{bmatrix} \\
&= \begin{bmatrix} v^*x & \bigstar & \cdots & \bigstar \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.
\end{aligned}$$

Therefore, the similarity

$$S^{-1}(A + xv^*)S = \begin{bmatrix} \lambda + v^*x & \bigstar & \cdots & \bigstar \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bigstar \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix},$$

reveals both the eigenvalues of $A + xv^*$ and their multiplicities.

A new alternative proof that the eigenvalues of $A(c)$ are $\lambda, c\lambda_2, \ldots, c\lambda_n$, only based on polynomial identities, is proposed below.

For any n–by–k complex matrices Z and W with $n \geq k$, the n eigenvalues of ZW^* are the k eigenvalues of W^*Z together with $n - -k$ zero eigenvalues [24, Theorem 1.3.20]. In particular, for any vectors $z, w \in \mathbb{C}^n$ the n eigenvalues of zw^* are $w^*z, 0, \ldots, 0$, so the n eigenvalues of $I + zw^*$ are $1 + w^*z, 1, \ldots, 1$. It follows that $\det(I + zw^*) = 1 + w^*z$.

Since $(tI - cA)x = (t - c\lambda)x$, we have $(tI - cA)^{-1}x = (t - c\lambda)^{-1}x$ for any $t \neq c\lambda$. For any $z \in \mathbb{C}^n$ and for $t \neq c\lambda$, compute

$$\begin{aligned}
p_{cA+xz^*}(t) &= \det(tI - (cA + xz^*)) \\
&= \det((tI - cA) - xz^*) \\
&= \det(tI - cA)\det(I - (tI - cA)^{-1}xz^*) \\
&= p_{cA}(t)\det(I - (t - c\lambda)^{-1}xz^*) \\
&= p_{cA}(t)\left(1 - (t - c\lambda)^{-1}z^*x\right) \\
&= \frac{p_{cA}(t)\,(t - c\lambda - z^*x)}{t - c\lambda}.
\end{aligned}$$

Thus, for any $z \in \mathbb{C}^n$ we have the polynomial identity

$$(t - c\lambda)p_{cA+xz^*}(t) = (t - (c\lambda + z^*x))p_{cA}(t), \tag{1.2}$$

where it is again legal to have $t = c\lambda$ by continuity arguments. The zeroes of the left–hand side are $c\lambda$ together with the eigenvalues of $cA + xz^*$; the zeroes of the right–hand side are $c\lambda + z^*x, c\lambda, c\lambda_2, \ldots, c\lambda_n$. It follows that the eigenvalues of $cA + xz^*$ are $c\lambda + z^*x, c\lambda_2, \ldots, c\lambda_n$.

Now set $z = (1 - \overline{c})\overline{\lambda}v$, use the condition $v^*x = 1$, and conclude that the eigenvalues of $A(c)$ are $\lambda, c\lambda_2, \ldots, c\lambda_n$ for any $c \in \mathbb{C}$.

Finally, it is worth mentioning a two–lines proof of Brauer's theorem due to Iannazzo [26] which could be considered a special case of a proof trick used in the functional formulation of the shift [6][Section 3.2], in a structured Markov chains context. Based on the matrix–polynomial identity and $Axv^* = \lambda xv^*$

$$(A + xv^* - \mu I)(\mu - \lambda)I = (A - \mu I)((\mu - \lambda)I - xv^*),$$

by taking the determinant of both members and using the formula for the characteristic polynomial of a dyad, it holds that

$$\begin{aligned} p_{A+xv^*}(\mu)(\mu - \lambda)^n &= (-1)^n p_A(\mu) p_{x^*v}(\mu - \lambda) \\ &= (-1)^n p_A(\mu)(\mu - \lambda)^{n-1}(\mu - \lambda - v^*x). \end{aligned}$$

The unique factorization theorem for polynomials achieves the proof.

It is worthwhile to remark that the interest of Iannazzo for Brauer's theorem does not come from the Google matrix, but from fast Markov chains computations, Riccati matrix equations etc. See [5] and references reported therein.

1.3 Complete principle of biorthogonality and Jordan blocks

Brauer used the basic principle of biorthogonality to analyze the *eigenvalues* of $A + xv^*$. We now want to analyze the *Jordan blocks* of $A + xv^*$.

The basic principle of biorthogonality is silent about what happens when $\lambda = \mu$. In that event, there are three possibilities: (i) $y^*x = 0$ (we can normalize so that $x^*x = y^*y = 1$); (ii) $y^*x \neq 0$ (we can normalize so that $y^*x = 1$); or (iii) $x = \alpha y$ (we can normalize so that $x = y$ and $x^*x = 1$). The following *complete principle of biorthogonality* addresses all the possibilities and describes reduced forms for A that can be achieved in each case.

1.3.1 Theorem. *Let A be an n–by–n complex matrix and let x and y be nonzero complex vectors such that $Ax = \lambda x$ and $y^*A = \mu y^*$.*

(a) Suppose that $\lambda \neq \mu$ and $x^*x = y^*y = 1$. Then $y^*x = 0$. Let $U = [x\ y\ U_1]$, in which the columns of U_1 are any given orthonormal basis for the orthogonal complement of x and y. Then U is unitary and

$$U^*AU = \begin{bmatrix} \lambda & \bigstar & \bigstar \\ 0 & \mu & 0 \\ 0 & \bigstar & B \end{bmatrix}, \quad B = U_1^*AU_1 \text{ is } (n-2)\text{–by–}(n-2).$$

(b) Suppose that $\lambda = \mu$, $y^*x = 0$, and $x^*x = y^*y = 1$. Let $U = [x\ y\ U_1]$, in which the columns of U_1 are any given orthonormal basis for the orthogonal complement of x and y. Then U is unitary, the algebraic multiplicity of λ is at least two, and

$$U^*AU = \begin{bmatrix} \lambda & \bigstar & \bigstar \\ 0 & \lambda & 0 \\ 0 & \bigstar & B \end{bmatrix}, \quad B = U_1^*AU_1 \text{ is } (n-2)\text{–by–}(n-2).$$

(c) Suppose that $\lambda = \mu$ and $y^*x = 1$. Let $S = [x\ S_1]$, in which the columns of S_1 are any given basis for the orthogonal complement of y. Then S is nonsingular, $(S^{-1})^* = [y\ Z_1]$, the columns of Z_1 are a basis for the orthogonal complement of x, and

$$S^{-1}AS = \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}, \quad B = Z_1^*AS_1 \text{ is } (n-1)\text{–by–}(n-1). \tag{1.3}$$

(d) Suppose that $\lambda = \mu$, $x = y$, and $x^*x = 1$. Let $U = [x\ U_1]$, in which the columns of U_1 are any given orthonormal basis for the orthogonal complement of x. Then U is unitary and

$$U^*AU = \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}, \quad B = U_1^*AU_1 \text{ is } (n-1)\text{–by–}(n-1). \tag{1.4}$$

Proof. (a) Lemma 1.2.1 ensures that x and y are orthogonal. Let $U = [x\, y\, U_1]$, in which the columns of U_1 are a given orthonormal basis for the orthogonal complement of x and y. The n columns of U are an orthonormal set, so U is unitary. Compute the unitary similarity

$$\begin{aligned} U^*AU &= \begin{bmatrix} x^* \\ y^* \\ U_1^* \end{bmatrix} A[x\, y\, U_1] = \begin{bmatrix} x^*Ax & x^*Ay & x^*AU_1 \\ y^*Ax & y^*Ay & y^*AU_1 \\ U_1^*Ax & U_1^*Ay & U_1^*AU_1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda x^*x & x^*Ay & x^*AU_1 \\ \lambda y^*x & \mu y^*y & \mu y^*U_1 \\ \lambda U_1^*x & U_1^*Ay & U_1^*AU_1 \end{bmatrix} = \begin{bmatrix} \lambda & \bigstar & \bigstar \\ 0 & \mu & 0 \\ 0 & \bigstar & U_1^*AU_1 \end{bmatrix}. \end{aligned}$$

(b) As in (a), construct a unitary matrix $U = [x\, y\, U_1]$, in which the columns of U_1 are a given orthonormal basis for the orthogonal complement of x and y. The reduced form of A under unitary similarity via U is the same as in (a), but with $\lambda = \mu$. The characteristic polynomial of A is

$$p_A(t) = \det(tI - A) = \det \begin{bmatrix} t-\lambda & \bigstar & \bigstar \\ 0 & t-\lambda & 0 \\ 0 & \bigstar & tI - B \end{bmatrix}$$

A Laplace expansion by minors down the first column gives

$$p_A(t) = (t-\lambda)\det \begin{bmatrix} t-\lambda & 0 \\ \bigstar & tI-B \end{bmatrix}.$$

Finally, a Laplace expansion by minors across the first row gives

$$p_A(t) = (t-\lambda)^2 \det(tI-B) = (t-\lambda)^2\, p_B(t),$$

so λ is a zero of $p_A(t)$ with multiplicity at least two.

(c) Let the columns of S_1 be a given basis for the orthogonal complement of y and let $S = [x\, S_1]$. The columns of S_1 are linearly independent, so S is singular only if x is a linear combination of the columns of S_1, that is, only if $x = S_1\xi$ for some vector ξ. But then $1 = y^*x = y^*S_1\xi = 0\xi = 0$. This contradiction shows that S is nonsingular. Partition $(S^{-1})^* = [\eta\, Z_1]$ and compute

$$I = S^{-1}S = \begin{bmatrix} \eta^* \\ Z_1^* \end{bmatrix} \begin{bmatrix} x & S_1 \end{bmatrix} = \begin{bmatrix} \eta^*x & \eta^*S_1 \\ Z_1^*x & Z_1^*S_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \end{bmatrix}. \tag{1.5}$$

Thus, the $n-1$ columns of Z_1, necessarily linearly independent, are orthogonal to x, so they are a basis for the orthogonal complement of x. Also, $\eta^* S_1 = 0$ means that η is orthogonal to the orthogonal complement of y, so $\eta = \alpha y$. But $1 = \eta^* x = (\alpha y)^* x = \bar{\alpha} y^* x = \bar{\alpha}$, so $\alpha = 1$ and $\eta = y$. Finally, compute the similarity

$$\begin{aligned} S^{-1}AS &= \begin{bmatrix} y^* \\ Z_1^* \end{bmatrix} A \begin{bmatrix} x & S_1 \end{bmatrix} = \begin{bmatrix} y^*Ax & y^*AS_1 \\ Z_1^*Ax & Z_1^*AS_1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda y^*x & \lambda y^*S_1 \\ \lambda Z_1^*x & Z_1^*AS_1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & Z_1^*AS_1 \end{bmatrix}. \end{aligned}$$

(d) Let the columns of U_1 be a given orthonormal basis for the orthogonal complement of x. Then the n columns of $U = [x\ U_1]$ are an orthonormal set, so U is unitary. Compute the unitary similarity

$$\begin{aligned} U^*AU &= \begin{bmatrix} x^* \\ U_1^* \end{bmatrix} A \begin{bmatrix} x & U_1 \end{bmatrix} = \begin{bmatrix} x^*Ax & x^*AU_1 \\ U_1^*Ax & U_1^*AU_1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda x^*x & \lambda x^*U_1 \\ \lambda U_1^*x & U_1^*AU_1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & U_1^*AU_1 \end{bmatrix}. \end{aligned}$$

□

We now use the complete principle of biorthogonality to establish an analog of Brauer's Theorem 1.2.2 for Jordan blocks.

1.3.2 Theorem. *Let A be an n–by–n complex matrix. Let $\lambda, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A, and let x and y be nonzero complex vectors such that $Ax = \lambda x$ and $y^*A = \lambda y^*$. Assume that $y^*x = 1$. Then the Jordan canonical form of A is*

$$[\lambda] \oplus J_{n_1}(\nu_1) \oplus \cdots \oplus J_{n_k}(\nu_k)$$

for some positive integers $k, n_1, \ldots, n_k$ and some set of eigenvalues $\{\nu_1, \ldots, \nu_k\} \subseteq \{\lambda_2, \ldots, \lambda_n\}$. For any complex vector v such that $\lambda + v^*x \neq \lambda_j$ for each $j = 2, \ldots, n$, the Jordan canonical form of $A + xv^*$ is

$$[\lambda + v^*x] \oplus J_{n_1}(\nu_1) \oplus \cdots \oplus J_{n_k}(\nu_k). \tag{1.6}$$

Proof. The hypotheses and Theorem 1.3.1(c) ensure that

$$S^{-1}AS = \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} \tag{1.7}$$

for some nonsingular S of the form $S = [x\ S_1]$, so that $S^{-1}x = e_1$. The eigenvalues of B are $\lambda_2, \ldots, \lambda_n$; let

$$J_{n_1}(\nu_1) \oplus \cdots \oplus J_{n_k}(\nu_k)$$

be the Jordan canonical form of B. Just as in Reams' proof of Brauer's Theorem, we have

$$S^{-1}(xv^*)S = \left(S^{-1}x\right)(v^*S) = e_1 \begin{bmatrix} v^*x & v^*S_1 \end{bmatrix} = \begin{bmatrix} v^*x & w^* \\ 0 & 0 \end{bmatrix}, \tag{1.8}$$

in which we set $w^* := v^*S_1$. Combining the similarities (1.7) and (1.8), we see that

$$S^{-1}(A + xv^*)S = \begin{bmatrix} \lambda + v^*x & w^* \\ 0 & B \end{bmatrix}.$$

Now let ξ be any given $(n-1)$–vector, verify that

$$\begin{bmatrix} 1 & \xi^* \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\xi^* \\ 0 & I \end{bmatrix},$$

and compute the similarity

$$\begin{bmatrix} 1 & -\xi^* \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda + v^*x & w^* \\ 0 & B \end{bmatrix} \begin{bmatrix} 1 & \xi^* \\ 0 & I \end{bmatrix} = \begin{bmatrix} \lambda + v^*x & w^* + \xi^*((\lambda + v^*x)I - B) \\ 0 & B \end{bmatrix}.$$

We have assumed that $\lambda + v^*x$ is not an eigenvalue of B, so we may take

$$\xi^* := -w^*((\lambda + v^*x)I - B)^{-1},$$

in which case $w^* + \xi^*((\lambda + v^*x)I - B) = 0$ and $A + xv^*$ is revealed to be similar to

$$\begin{bmatrix} \lambda + v^*x & 0 \\ 0 & B \end{bmatrix}.$$

Thus, the Jordan canonical form of $A + xv^*$ is (1.6): the direct sum of $[\lambda + v^*x]$ and the Jordan canonical form of B. $\square$

The following result strengthens the conclusion of Corollary 1.2.3 to describe not only the eigenvalues of $A(c)$, but also its Jordan blocks.

1.3.3 Corollary. *Let A be an n–by–n complex matrix. Let $\lambda, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A; let x, y, and v be nonzero complex vectors such that $Ax = \lambda x$, $y^*A = \lambda y^*$, and $v^*x = 1$; and let $A(c) = cA + (1-c)\lambda xv^*$. Assume that $y^*x = 1$ and integers $k, n_1, \ldots, n_k$ and the set $\{\nu_1, \ldots, \nu_k\}$ are defined as in the previous Theorem. Let the Jordan canonical form of A be*

$$[\lambda] \oplus J_{n_1}(\nu_1) \oplus \cdots \oplus J_{n_k}(\nu_k).$$

Then for any nonzero complex number c such that

$$c\lambda_j \neq \lambda \text{ for each } j = 2, \ldots, n, \tag{1.9}$$

the Jordan canonical form of $A(c)$ is

$$[\lambda] \oplus J_{n_1}(c\nu_1) \oplus \cdots \oplus J_{n_k}(c\nu_k).$$

Proof. We proceed as in the proof of Corollary 1.2.3. In the statement of Theorem 1.3.2, replace A and v, respectively, by cA and $(1-\bar{c})\bar{\lambda}v$, respectively. For any c, cA is similar to

$$[c\lambda] \oplus cJ_{n_1}(\nu_1) \oplus \cdots \oplus cJ_{n_k}(\nu_k),$$

but if $c \neq 0$, we can say more: this direct sum is similar to

$$[c\lambda] \oplus J_{n_1}(c\nu_1) \oplus \cdots \oplus J_{n_k}(c\nu_k).$$

Moreover, x is an eigenvector of cA associated with the eigenvalue $c\lambda$, the remaining eigenvalues of cA are $c\lambda_2, \ldots, c\lambda_n$, and

$$c\lambda + ((1-\bar{c})\bar{\lambda}v)^*x = c\lambda + (1-c)\lambda v^*x = c\lambda + (1-c)\lambda = \lambda.$$

Thus, our assumption (1.9) and Theorem 1.3.2 ensure that the Jordan canonical form of

$$cA + x((1-\bar{c})\bar{\lambda}v)^* = cA + (1-c)\lambda xv^* = A(c)$$

is

$$[\lambda] \oplus J_{n_1}(c\nu_1) \oplus \cdots \oplus J_{n_k}(c\nu_k).$$

□

In the above analysis, often the matrix B is determined only up to similarity. If convenient, we can take B to be a Jordan canonical form, upper triangular, a real Jordan form (if A is real), a Schur canonical form, etc. Perhaps this flexibility can be exploited to achieve a computational advantage.

Finally we stress a pleasant contrast between Corollary 1.2.3 and Corollary 1.3.3. In Corollary 1.2.3 the hypothesis is weaker than that of Corollary 1.3.3, and of course a weaker conclusion is obtained. However, Corollary 1.2.3 is of independent interest, since it gives a broader context for the famous eigenvalue properties of the Google matrix perturbation: for instance, similar problems appear and Corollary 1.2.3 is useful in the context of iterative solvers for algebraic Riccati equation, for accelerating the convergence of cyclic reduction based algorithms (see [7, 5] and references therein and [29] for further applications of mathematical physics).

1.4 The normalized left λ–eigenvector of $A(c)$

If $\lambda \neq 0$, Corollary 1.2.3 ensures that it is a simple eigenvalue of $A(c)$ for all but finitely many values of c. We would like to have an explicit expression for its associated left eigenvector $y(c)$, normalized so that $y(c)^* x = 1$.

1.4.1 Theorem. *Let A be an n–by–n complex matrix. Let $\lambda, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A; let $\mu_1, \dots, \mu_d$ be the nonzero eigenvalues of A that are different from λ; let x and v be nonzero complex vectors such that $Ax = \lambda x$ and $v^* x = 1$; and let $A(c) = cA + (1-c)\lambda x v^*$. Assume that $\lambda \neq 0$.*
(i) Suppose that there is a complex vector y such that $y^* A = \lambda y^*$ and $y^* x = 1$. Assume that $c\lambda_j \neq \lambda$ for each $j = 2, \dots, n$. Let S_1, Z_1, and B be defined as in Theorem 1.3.1(c). Then λ is not an eigenvalue of cB. Define the vector $y(c)$ by

$$y(c)^* = y^* + (1-c)\lambda v^* S_1 (\lambda I_{n-1} - cB)^{-1} Z_1^*. \tag{1.10}$$

Then $y(c)$ is the only vector that satisfies the conditions

$$y(c)^* A(c) = \lambda y(c)^* \text{ and } y(c)^* x = 1. \tag{1.11}$$

If λ is a simple eigenvalue of A, then it is not an eigenvalue of B.
(ii) Suppose that λ is a semisimple eigenvalue of A with multiplicity $m \geq 2$

and suppose that

$$c\mu_j \neq \lambda \text{ for each } j = 1,\ldots,d. \tag{1.12}$$

Let $S = [X\ S_2]$ *be any nonsingular matrix such that* X *has* m *columns and*

$$S^{-1}AS = \begin{bmatrix} \lambda I_m & 0 \\ 0 & E \end{bmatrix}, \quad E \text{ is } (n-m)\text{–by–}(n-m). \tag{1.13}$$

Then λ *is not an eigenvalue of* cE *or* E*. Partition* $(S^{-1})^* = [Y\ Z_2]$*, in which* Y *has* m *columns. Then* $AX = \lambda X$*,* $Y^*A = \lambda Y^*$*, and* $Y^*X = I_m$*. Moreover, the columns of* X *may be chosen to be any* m *linearly independent right* λ*–eigenvectors of* A*, and*

$$XY^* = I - (\lambda I - A)(\lambda I - A)^D \tag{1.14}$$

is a projection that is determined uniquely by A *and* λ*, regardless of the choice of columns of* X*. Define the vector* $y(c)$ *by*

$$y(c)^* = v^*XY^* + (1-c)\lambda v^* S_2(\lambda I_{n-m} - cE)^{-1} Z_2^*. \tag{1.15}$$

Then $y(c)$ *satisfies the conditions (1.11); if, in addition,* $c \neq 1$*, then* $y(c)$ *is the only vector that satisfies these conditions. If both* A *and* λ *are real, then* XY^* *is a real projection.*

(iii) Suppose that λ *is a semisimple eigenvalue of* A *with multiplicity* m*. Let* K *be a given compact complex set that does not contain any of the points* $\lambda\mu_1^{-1},\ldots,\lambda\mu_d^{-1}$*. Let* $\tilde{c}$ *and* c *be distinct points in* K*. If* $m \geq 2$*, let* $y(\cdot)^*$ *be defined by (1.15). Then*

$$\frac{y(\tilde{c})^* - y(c)^*}{\tilde{c} - c} = \lambda v^* S_2(\tilde{c}E - \lambda I)^{-1}(E - \lambda I)(cE - \lambda I)^{-1} Z_2^*; \tag{1.16}$$

the derivative of $y(c)$ *is*

$$y'(c)^* = \lambda v^* S_2(cE - \lambda I)^{-2}(E - \lambda I) Z_2^*; \tag{1.17}$$

the derivative of $y(c)^*$ *at* $c = 0$ *is*

$$y'(0)^* = \lambda^{-1} v^* S_2(E - \lambda I) Z_2^* = \lambda^{-1} v^*(A - \lambda I); \tag{1.18}$$

and the derivative of $y(c)^*$ at $c = 1$ is

$$y'(1)^* = \lambda v^* S_2(E - \lambda I)^{-1} Z_2^* = \lambda v^*(A - \lambda I)^D. \tag{1.19}$$

If $m = 1$ and $y(\cdot)$ is defined by (1.10), then the four preceding identities are correct if we replace E with B, S_2 with S_1, and Z_2 with Z_1. Finally, independently of $m \geq 1$, for each given vector norm $\|\cdot\|$ there is a positive constant M (depending on A, λ, v, and K) such that

$$\|y(\tilde{c}) - y(c)\| \leq M|\widetilde{c} - c| \text{ for all } \widetilde{c}, c \in K. \tag{1.20}$$

Proof. (i) The similarity (1.3) shows that the eigenvalues of B are $\lambda_2, \ldots, \lambda_n$, so our assumption that $\lambda \neq c\lambda_j$ for all $j = 2, \ldots, n$ ensures that λ is not an eigenvalue of cB. If λ is an eigenvalue of B it must have multiplicity at least two as an eigenvalue of A, so if it is a simple eigenvalue of A it is not an eigenvalue of B. The vector $y(c)$ defined by (1.10) satisfies the condition $y(c)^* x = 1$ because $y^* x = 1$ and $Z_1^* x = 0$. To show that it is a left λ–eigenvector of $A(c)$, we begin by combining (1.7) and (1.8):

$$S^{-1}(cA + (1-c)\lambda x v^*)S = \begin{bmatrix} \lambda & (1-c)\lambda v^* S_1 \\ 0 & cB \end{bmatrix}. \tag{1.21}$$

A calculation verifies that the vector $\eta(c)$ defined by

$$\eta(c)^* = [1 \quad (1-c)\lambda v^* S_1(\lambda I_{n-1} - cB)^{-1}]$$

is a left λ–eigenvector of the matrix in (1.21) and $\eta(c)^* e_1 = 1$; if $c \neq 1$, it is the only such vector. Therefore, the vector $y(c)$ defined by

$$\begin{aligned} y(c)^* &= \eta(c)^* S^{-1} = [1 \quad (1-c)\lambda v^* S_1(\lambda I_{n-1} - cB)^{-1}] \begin{bmatrix} y^* \\ Z_1^* \end{bmatrix} \\ &= y^* + (1-c)\lambda v^* S_1(\lambda I_{n-1} - cB)^{-1} Z_1^* \end{aligned}$$

is a normalized left λ–eigenvector of $A(c)$, and it is the only vector that satisfies the conditions (1.11).

(ii) Let D denote the block diagonal matrix in (1.13), and let S be any nonsingular matrix such that $S^{-1}AS = D$. Partition $S = [X \ S_2]$ and $(S^{-1})^* = [Y \ Z_2]$, in which X and Y have m columns. Then

$$[AX \ AS_2] = AS = SD = [\lambda X \ S_2 D],$$

and

$$\begin{bmatrix} Y^*A \\ Z_2^*A \end{bmatrix} = S^{-1}A = DS^{-1} = \begin{bmatrix} \lambda Y^* \\ EZ_2^* \end{bmatrix},$$

which tells us that the columns of X are a linearly independent set of right λ–eigenvectors of A and the columns of Y are a linearly independent set of left λ–eigenvectors of A. The identity $S^{-1}S = I$ tells us that $Y^*X = I_m$ and hence that $X^*Y = (Y^*X)^* = I_m^* = I_m$.

Now let R be any given nonsingular m–by–m matrix, let $\hat{S} = [XR \;\; S_2] := [\hat{X} \;\; S_2]$, partition $(\hat{S}^{-1})^* = [\hat{Y} \;\; Z_2]$, compute $(\hat{S}^{-1})^* = [Y(R^{-1})^* \;\; Z_2]$, and notice that $\hat{Y}\hat{X}^* = YX^*$. We draw two conclusions from these observations: (1) We are free to let the columns of X be *any* linearly independent set of right λ–eigenvectors of A; and (2) Regardless of the choice of columns of X, the product YX^* remains the same. Moreover, $(YX^*)^2 = Y(X^*Y)X^* = YI_mX^* = YX^*$, so YX^* (and hence also XY^*) is a projection.

This second conclusion also follows from a useful representation for XY^*. We have

$$\lambda I - A = S \begin{bmatrix} 0 & 0 \\ 0 & \lambda I - E \end{bmatrix} S^{-1} \text{ and } (\lambda I - A)^D = S \begin{bmatrix} 0 & 0 \\ 0 & (\lambda I - E)^{-1} \end{bmatrix} S^{-1},$$

and hence

$$I - (\lambda I - A)(\lambda I - A)^D = I - S \begin{bmatrix} 0 & 0 \\ 0 & I_{n-m} \end{bmatrix} S^{-1} = [X \;\; S_2] \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y^* \\ Z_2^* \end{bmatrix} = XY^*.$$

Let the first column of X be the given λ–eigenvector x such that $v^*x = 1$, and write $X = [x \; \widetilde{X}]$. Then x is the first column of S, so $S^{-1}x = e_1$ and

$$v^*S = [v^*X \;\; v^*S_2] = [v^*x \;\; v^*\widetilde{X} \;\; v^*S_2] = [1 \;\; v^*\widetilde{X} \;\; v^*S_2].$$

Thus,

$$S^{-1}(xv^*)S = (S^{-1}x)(v^*S) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & v^*\widetilde{X} & v^*S_2 \end{bmatrix} = \begin{bmatrix} 1 & v^*\widetilde{X} & v^*S_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.22}$$

and so

$$S^{-1}\left(cA+(1-c)\lambda xv^*\right)S=\begin{bmatrix}\lambda & (1-c)\lambda v^*\widetilde{X} & (1-c)\lambda v^*S_2\\ 0 & c\lambda I_{m-1} & 0\\ 0 & 0 & cE\end{bmatrix}. \qquad (1.23)$$

The assumption (1.12) (which is satisfied for $c=1$) ensures that λ is not an eigenvalue of cE, and a calculation verifies that $\eta(c)$ defined by

$$\begin{aligned}\eta(c)^* &= [1 \;\; v^*\widetilde{X} \;\; (1-c)\lambda v^*S_2(\lambda I-cE)^{-1}]\\ &= [v^*X \;\; (1-c)\lambda v^*S_2(\lambda I-cE)^{-1}]\end{aligned}$$

is a left λ–eigenvector of the matrix in (1.23) and $\eta(c)^*e_1=1$; if $c\neq 1$ it is the unique such vector. Therefore, $y(c)$ defined by

$$\begin{aligned}y(c)^* &= \eta(c)^*S^{-1}=\begin{bmatrix} v^*X & (1-c)\lambda v^*S_2\left(\lambda I-cE\right)^{-1}\end{bmatrix}\begin{bmatrix}Y^*\\ Z_2^*\end{bmatrix}\\ &= v^*XY^*+(1-c)\lambda v^*S_2(\lambda I-cE)^{-1}Z_2^*\end{aligned}$$

satisfies the conditions (1.11); if $c\neq 1$ it is the only vector that satisfies these conditions.

If A and λ are real, the matrix $S=[X\ S_2]$ that gives the reduced form (1.13) may be taken to be real (one may reduce to the real Jordan form, for example [24, Theorem 3.4.5]). Then $(S^{-1})^*=[Y\ Z_2]$ is real, so the uniquely determined product XY^* must always be real, regardless of the choice of X.

(iii) Using the identity $\alpha R^{-1}-\beta T^{-1}=R^{-1}(\alpha T-\beta R)T^{-1}$, we compute

$$\begin{aligned}y(\tilde{c})^*-y(c)^* &= \lambda v^*S_2((1-\tilde{c})(\lambda I-\tilde{c}E)^{-1}-(1-c)(\lambda I-cE)^{-1})Z_2^*\\ &= (\tilde{c}-c)\lambda v^*S_2(\tilde{c}E-\lambda I)^{-1}(E-\lambda I)(cE-\lambda I)^{-1}Z_2^*.\end{aligned}$$

This identity verifies (1.16). One obtains (1.17) by letting $\tilde{c}\to c$; (1.18) and (1.19) follow by setting $c=1$ and $c=0$, respectively. The bound (1.20) follows from taking the norm of both sides of (1.16) and observing that the right–hand side is a continuous function on a compact set, so it is bounded. □

The vector function $y(c)$ defined by (1.15) is a complex analytic function at all but finitely many points in the complex plane, provided that λ is a nonzero semisimple eigenvalue of A. The points $c=0$ and $c=1$ are of special interest.

- The condition (1.12) is satisfied for all c such that $|c|<\min\{|\lambda\mu_j^{-1}|: j=1,\ldots,d\}$. Thus, $y(c)$ is analytic in an open neighborhood of $c=0$ and can be represented there by a Maclaurin series obtained from (1.15) by expanding $(\lambda I_{n-m}-cE)^{-1}$ as a power series in c:

$$\begin{aligned} y(c)^* &= v^*\left(I+\sum_{k=1}^{\infty}\lambda^{-k}\left(S_2(E-\lambda I)E^{k-1}Z_2^*\right)c^k\right) \\ &= v^*\left(I+\sum_{k=1}^{\infty}\lambda^{-k}\left((A-\lambda I)A^{k-1}\right)c^k\right). \end{aligned} \tag{1.24}$$

  This representation reveals all of the derivatives of $y(c)$ at $c=0$.

- The condition (1.12) is also satisfied for all c such that $|c-1|<\min\{|\lambda\mu_j^{-1}-1|: j=1,\ldots,d\}$. Thus, $y(c)$ is analytic in an open neighborhood of $c=1$. If we let $\gamma=c-1$, use (1.15), and expand

$$(\lambda I-cE)^{-1}=(\lambda I-E)^{-1}\left(I-\gamma E(\lambda I-E)^{-1}\right)^{-1}$$

  as a power series in γ, we obtain

$$y(\gamma+1)^*=v^*\left(XY^*-\lambda\sum_{k=1}^{\infty}\left(S_2(\lambda I-E)^{-k}E^{k-1}Z_2^*\right)\gamma^k\right). \tag{1.25}$$

  This series reveals all the derivatives of $y(c)$ at $c=1$. We can use the Drazin inverse to write this series as

$$y(\gamma+1)^*=v^*\left(XY^*-\lambda\sum_{k=1}^{\infty}\left(((\lambda I-A)^D)^kA^{k-1}\right)\gamma^k\right). \tag{1.26}$$

  In particular, the first derivative is

$$y'(1)^*=\lambda v^*S_2(E-\lambda I)^{-1}Z_2^*=\lambda v^*(A-\lambda I)^D. \tag{1.27}$$

1.4.1 The behavior of $y(c)$ as $c\to 1$

We are interested in the behavior of the left eigenvector $y(c)$ defined by (1.10) as $c\to 1$ in the complex plane for $\lambda\neq 0$.

The existence of the limit as c tends to 1 via any complex path requires the eigenvalue $\lambda \neq 0$ to be semisimple as the following two examples show.

Example 1. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \ \lambda = 1, \ x = y = e_1, \ v^* = \begin{bmatrix} 1 & \alpha & \beta \end{bmatrix}$$

and the vector $y(c)$ defined by

$$y(c)^* = \begin{bmatrix} 1 & \alpha & \frac{(c-1)\beta}{2c-1} \end{bmatrix}.$$

$y(c)^*$ is the normalized left eigenvector of

$$cA + (1-c)\lambda x v^* = \begin{bmatrix} \lambda + c(1-\lambda) & \lambda(1-c)\alpha & \lambda(1-c)\beta \\ 0 & c & 0 \\ 0 & 0 & 2c \end{bmatrix}$$

associated with the eigenvalue $\lambda = 1$. Moreover,

$$\lim_{c \to 1} y(c)^* = \begin{bmatrix} 1 & \alpha & 0 \end{bmatrix}.$$

Although $\lambda = 1$ is not a simple eigenvalue of A, it is semisimple.

Example 2. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ \lambda = 1, \ x = y = e_1, \ v^* = \begin{bmatrix} 1 & \alpha & \beta \end{bmatrix}$$

and the vector $y(c)$ defined by

$$y(c)^* = \begin{bmatrix} 1 & \alpha & \beta + \frac{c\alpha}{1-c} \end{bmatrix}.$$

$y(c)^*$ is the normalized left eigenvector of

$$cA + (1-c)\lambda x v^* = \begin{bmatrix} \lambda + c(1-\lambda) & \lambda(1-c)\alpha & \lambda(1-c)\beta \\ 0 & c & c \\ 0 & 0 & c \end{bmatrix}$$

associated with the eigenvalue $\lambda = 1$. However, $\lim_{c \to 1} y(c)^*$ does not exist unless $\alpha = 0$. In this case, $\lambda = 1$ is not semisimple.

As previously mentioned, when $\lambda \neq 0$, the essential hypothesis required to ensure that $\lim_{c\to 1} y(c)$ exists for all choices of v is that λ is semisimple. The following Theorem verifies this assertion and gives an explicit formula for the limit.

1.4.2 Theorem. *Let A be an n–by–n complex matrix with eigenvalues $\lambda, \lambda_2, \ldots, \lambda_n$. Suppose that λ is a nonzero semisimple eigenvalue of A with multiplicity $m \geq 1$; let x and v be given nonzero complex vectors such that $Ax = \lambda x$ and $v^*x = 1$; and let $A(c) = cA + (1-c)\lambda x v^*$. If $m = 1$, let y be the unique vector such that $y^*A = \lambda y$ and $y^*x = 1$. If $m > 1$, let $XY^* = I - (\lambda I - A)(\lambda I - A)^D$ be the projection defined in Theorem 1.4.1(ii). Then*

(i) For some $\varepsilon > 0$ and all complex c such that $0 < |c-1| < \varepsilon$, as well as for all complex c such that $\lambda \neq c\lambda_j$ for all $j = 2, \ldots, n$, the vector $y(c)$ defined by (1.10) when λ is simple, or by (1.15) when it is not, is the unique vector that satisfies $y(c)^*A(c) = \lambda y(c)^*$ and $y(c)^*x = 1$.

(ii) If λ is a simple eigenvalue of A, then $\lim_{c\to 1} y(c) = yx^*v = y$.

(iii) If $m > 1$, then

$$\lim_{c\to 1} y(c) = YX^*v = (XY^*)^*v. \tag{1.28}$$

Proof. (i) If λ and 0 are the only eigenvalues of A, then any positive value of ε will do. If the nonzero eigenvalues of A that are different from λ are $\mu_1, \ldots, \mu_d$, let

$$\varepsilon = \min\{|1-\lambda\mu_1^{-1}|, \ldots, |1-\lambda\mu_d^{-1}|\}.$$

Then the hypothesis (1.9) is satisfied and the assertion follows from Theorem 1.4.1. Since $y(c)$ is defined in a punctured open complex neighborhood of the point $c = 1$, it is reasonable to ask about the limit of $y(c)$ (as a function of the complex variable c) as $c \to 1$.

(ii) The assertion follows from (1.10) since λ is not an eigenvalue of B:

$$\begin{aligned}
\lim_{c\to 1} y^*(c) &= y^* + \lim_{c\to 1}\left((1-c)\lambda v^* S_1 (\lambda I - cB)^{-1} Z_1^*\right) \\
&= y^* + \lim_{c\to 1}(1-c)\cdot\lambda v^* S_1 \cdot \lim_{c\to 1}(\lambda I - cB)^{-1} Z_1^* \\
&= y^* + \left(0\cdot\lambda v^* S_1 (\lambda I - B)^{-1} Z_1^*\right) = y^* = v^*xy^*.
\end{aligned}$$

(iii) This assertion follows in the same way from (1.15):

$$\begin{aligned}
\lim_{c\to 1} y(c)^* &= v^*XY^* + \lim_{c\to 1}\big((1-c)\lambda v^* S_2(\lambda I_{n-m} - cE)^{-1}Z_2^*\big) \\
&= v^*XY^* + \lim_{c\to 1}(1-c)\cdot\lambda v^* S_2 \cdot \lim_{c\to 1}(\lambda I_{n-m} - cE)^{-1}Z_2^* \\
&= v^*XY^* + 0\cdot\lambda v^* S_2\cdot(\lambda I_{n-m} - E)^{-1}Z_2^* = v^*XY^*.
\end{aligned}$$

$\square$

Chapter 2

Google PageRanking

In this Chapter we have four main expository and research goals.

The first concerning a discussion on the model and on its adherence to the reality: a basic example presented in Section 2.2 is used to point out pathologies and limitations of the actual model and to propose some possible improvements.

Second we would like to understand the characteristics of the matrix $G(c)$ as a function of the complex parameter c (by completing the analysis in [39, 9]): we are interested in the eigenvalues and in the eigenvector structure, so that the analysis of canonical forms (Jordan, Schur etc.) is of prominent interest.

Third we would like to understand the behavior (regularity, expansions, limits, conditioning etc.) of the PageRank vector $y(c)$ as a function of c also for c close or equal to 1, and fourth we are interested in using the analytical characterization of $y(c)$ for computational purposes. In particular, it is known that for $c = 1$ the problem is ill–posed since there exist infinitely many left eigenvectors $y(1)$ of l^1 norm equal 1 and they form a convex set. On the other hand, for $c \in [0, 1)$, the solution exists and is unique, but the known algorithms become very slow when c is close to 1. Our interest is to compute $y(c)$ in these difficult cases, especially in the limit as c tends to 1.

The philosophical message that can be extracted from the latter three points is as follows: it has been said that the "PageRank problem is closely related to Markov chains" [13, p. 553]; however, framing the PageRank problem in the general setting of standard matrix–analytic properties of complex matrices can liberate one's imagination and stimulate novel approaches that might not be considered in the context of Markov chains.

This Chapter is organized as follows. In Section 2.1 we present the basic Google model, while in Section 2.2 we analyze it from a critical point of view and we propose improvements that prevent pathologic behaviors. In Section 2.3 we focus on the special case $A(c) = cA + (1-c)\lambda xv^*$ in which A is the basic Google matrix G, $\lambda = 1$, $x = e$ while v and c are complex; in this respect Section 2.4 is devoted to a comparison with the explicit formulae of $y(c)$ and of the Jordan form in [39], to a detailed analysis of the conditioning of $y(c)$ also for $c = 1$, and of the eigenvector structure of $G(c)$. In Section 2.5, we propose few algorithmic ideas for computing PageRank. The first exploits properties of $G(c)$ as a function of the complex variable c, especially in the unit open disk and in a proper disk centered at $c = 1$, while the second is based on a proper shift of the matrix G. Furthermore some remarks on the interpretation of the vector $\widetilde{y}$, the limit value of $y(c)$ as c tends to 1, are given. Section 2.6 mentions some prior work and Section 2.7 is devoted to concluding remarks and future work.

2.1 Google PageRanking model

As customary in the literature (see e.g. [31]), the Web can be regarded as a huge directed graph whose n nodes are all the Web pages and whose edges are constituted by all the direct links between pages. If $\deg(i)$ indicates the cardinality of the pages different from i which are reached by a direct link from page i, the simplest Google matrix G is defined as $G_{i,j} = 1/\deg(i)$ if $\deg(i) > 0$ and there exists a link in the Web from page i to a certain page $j \neq i$. In the case where $\deg(i) = 0$ (the so–called dangling nodes), we set $G_{i,j} = 1/n$ where n is the size of the matrix, i.e., the cardinality of all the Web pages. This definition is a model for the behavior of a generic Web user: if the user is visiting page i with $\deg(i) > 0$, then with probability $1/\deg(i)$ he/she will move to one of the pages $j \neq i$ linked by i; if i is a dangling node, i.e., it has no links, then the user will make just a random choice with uniform distribution $1/n$. The basic PageRank is an n sized vector which gives a measure of the importance of every page in the Web and this notion of importance of a given page is measured according to the limit probability that a generic user reaches that page asymptotically, i.e., after infinitely many clicks: this is the surfing model. On the other hand, we would like to have a more intrinsic and intuitive notion of importance or ranking of the Web pages. Indeed, taking inspiration from social sciences, the following ideas are quite natural:

- a page j is more important if there exists a page i referring to it,
- if i is a "very important page" and is referring j, then the importance of j is increased,
- if i is referring to many pages including $j \neq i$, i.e. $\deg(i)$ is large, then this adds little importance to j.

It is worth mentioning that the idea contained in the above itemized sentences is exactly a quantification of the notion of VIP (very important person) appearing in social sciences or, quite equivalently, according to a famous sentence of the PopArt master Andy Warhol "Don't pay attention to what they write about you. Just measure it in inches" (with several distinguished precursors "I don't care what you say about me, as long as you say something about me, and as long as you spell my name right" (George Cohan), "The only thing worse than being talked about is not being talked about" (Oscar Wilde), etc.). By the way, this basic observation shows the large potential of these researches in terms of the broad range of possible applications; see e.g. [4] for a recent study in the context of bibliometry.

Now we translate in formulae these concepts. More in detail, after a reasonable normalization, for every $j = 1, \ldots, n$, the importance $y[j]$ of page j is defined as follows

$$y[j] = \sum_{i \to j} \frac{y[i]}{\deg(i)}, \quad y[j] \geq 0, \qquad \sum_{i=1}^{n} y[i] = 1.$$

The definition is nice in principle and can be interpreted in matrix–vector terms as $y^T \hat{G} = y^T$, $y[j] \geq 0$, for all j, $\sum_{i=1}^{n} y[i] = 1$, where $\hat{G}_{i,j} = G_{i,j}$ if there exists in the Web a link from i to j and $\hat{G}_{i,j} = 0$ otherwise: G and $\hat{G}$ are the same with the exception of the management of dangling nodes. However, even by interpreting the above relations as an eigenvector problem with respect to the eigenvalue 1, either 1 may belong or may fail to belong to the spectrum of the resulting matrix. Explicit and very simple examples can be constructed: take e.g. the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

associated with a toy Web with only two nodes i, j with $i < j$ and a unique link from i to j; it is clear that the problem defined by $y^T \hat{G} = y^T$, $y[j] \geq 0$, for all j, $\sum_{i=1}^{n} y[i] = 1$ has no solution, since 1 is not in the spectrum of $\hat{G}$. The reason is again the presence of dangling nodes that in turn implies the existence of identically zero rows. Hence, for giving a solution to the above mathematical incongruence, we define

$$\begin{aligned} \deg^*(i) &= \deg(i), \quad \text{if} \;\; \deg(i) > 0, \\ \deg^*(i) &= n, \quad \text{if} \;\; \deg(i) = 0, \end{aligned} \tag{2.1}$$

and we correct accordingly the relations concerning $y[j]$ in the following way:

$$y[j] = \sum_{i \to j} \frac{y[i]}{\deg^*(i)}, \quad y[j] \geq 0, \qquad \sum_{i=1}^{n} y[i] = 1. \tag{2.2}$$

Putting the above relations in matrix terms, and introducing the l^1 norm of a real or complex vector w as $\|w\|_1 = \sum_{j=1}^{n} |w[j]|$, we have

$$y^T G = y^T, \quad y \geq 0, \quad \|y\|_1 = 1. \tag{2.3}$$

Interestingly enough, it should be observed that any vector y solution to (2.3) represents also a solution in the sense of the surfing model and vice versa. Therefore, in other words, with the above choice, there is an identification, which can be criticized between the surfing model and the definition of importance: in fact the definition of G referred to the dangling nodes is perfectly coherent in the surfing model, while is not justified at all when defining a notion of importance (see Section 2.2).

Now looking at (2.3), a basic PageRank is a nonnegative left eigenvector y of G associated with the dominating eigenvalue 1 normalized so that $\|y\|_1 =$

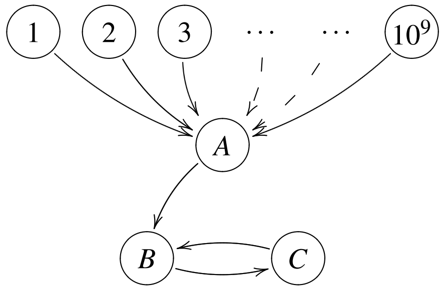
$y^T e = 1$, e being the vector of all ones (see e.g. [37, 28]). Since the matrix G is nonnegative and has row sum equal to 1 it is clear that a (canonical) right eigenvector related to 1 is e and that all the other eigenvalues are in modulus at most equal to 1.

Consequently, the good news is that a solution always exists; the bad news is that there might be multiple independent nonnegative solutions. And even if there is a unique solution, computing it by standard methods such as the power method [22] may fail, because G has one or more eigenvalues different from 1 that have modulus 1, see [23].

In fact, the structure of G is such that we have no guarantee for its aperiodicity and for its irreducibility: therefore the gap between 1 and the modulus of the second largest eigenvalue can be zero, see [23]. This means that the computation of the PageRank by the application of the standard power method (see e.g. [22]) to the matrix G^T (or one of its variations for our specific problem) is not convergent in general. A solution is found by considering a change in the model: given a value $c \in [0,1)$, from the basic Google matrix G we define the parametric Google matrix $G(c)$ as $cG+(1-c)ev^T$ with $v[i] \geq 0$, $\|v\|_1 = 1$. This change corresponds to the following user behavior: if the user is visiting page i, then the next move will be with probability c according to the rule described by the basic Google matrix G and with probability $1-c$ according to the rule described by v. We notice that this change is again meaningful in terms of the surfing model, but there is no clear interpretation in terms of notion of importance. Generally a value of *teleportation parameter* c as 0.85 is considered in the literature (see e.g. [28]). For $c \ll 1$, the good news is that the $y(c)$, i.e., the left dominating nonnegative eigenvector solution of (2.3) with $G = G(c)$, is unique and can be computed in a fast way since $G(c)$ has second eigenvalue whose modulus is dominated by c (the matrix $G(c)$ is now irreducible and primitive ref Section 3.1, see also [31] and references therein): therefore the convergence to $y(c)$ is such that the error at step k decays in the generic case as c^k. Of course the computation becomes slow if c is chosen close to 1 and there is no guarantee of convergence if $c = 1$.

2.2 Comments and proposals on the model

Let us start our discussion on the model, by considering in detail the following (extreme) example:



According to the classical algorithm in the ideal case (i.e. for $c = 1$, $y(1) = \widetilde{y}$) the page A has zero PageRank (as the 10^9 pages in the first row) and the importance is concentrated in B and C. In some sense the obtained ranking is against common sense since, given the topology of the graph, it is clear that page A should have a significant PageRank measure. This and other related pathologies need further investigations and this is the goal of the rest of this section in which we will suggest a revision of the PageRank model.

2.2.1 A monotonicity principle and the transient behavior

As already mentioned, according to the classical algorithm in the "ideal case", page A has zero ranking (as the 10^9 pages in the first row) and the importance is concentrated in B and C. This ranking is highly counter–intuitive and indeed wrong: if you are a leader of 10^9 people, you are really powerful no matter if any of your followers has low ranking, i.e., he/she is not important ...

Now suppose that C is deleted, i.e., the considered Web page is deleted: this can be also interpreted as merging B and C in a unique node that we can still call B. Then the rankings of any of 10^9 pages in the first row will move slightly from 0 to $1/(3(10^9+1))$; on the other hand the ranking of B goes down dramatically from $1/2$ to $1/3+1/(3(10^9+1))$ and A becomes really a leader moving from 0 to $1/3$. Again this sharp modification of the ranking is highly counter–intuitive and indeed wrong. At least, one would expect that the cumulative ranking of B and C equal to 1, before deletion of C, and the ranking of B alone after deletion of C should remain roughly speaking equal: a sort of monotonicity which is substantially violated by the actual model, which, on the other hand, induces an unmotivated discontinuous behavior in the solutions. In this respect, the original reason of such a pathology relies on the opposite extremal behavior of functions $\deg(\cdot)$ and $\deg^*(\cdot)$ in which a zero row is replaced by e^T/n. We notice that in the literature a wider idea has been considered by replacing e^T/n by any stochastic vector w^T: however the discontinuity in the model still remains and in the following different solutions to the problem are considered.

A strong and macroscopic evidence of the problems in the actual model is that for most of the nodes in real Web examples (what is called "core" in the literature) the ranking is zero. Indeed, only the use of values of the teleportation parameter c far away from 1 (e.g. 0.85) partly alleviates the problem (in Latin "ex falso quod libet" ..., is an expression capturing the fact that from something wrong anything can derive and, by coincidence, also good things...). In actuality, a basic error is the confusion between the notion of "importance" (PageRanking) and the stochastic model for surfing on the web. We can identify two critical points.

We have a somehow unnatural (wrong) treatment of dangling nodes: with the actual model, there is no monotonicity as the example of deletion of node C in the above graph shows. In a new model, the management of dangling nodes should be changed for insuring a sort of monotonicity.

The other substantial problem is that the transient effects are not taken into account. A user is on the Web for a finite number of clicks, at every visit. This implies that looking at the stationary vector, as the number of clicks tends to infinity, is just theoretical and far from reality. A new model has to incorporate the transient behavior (see also the functional approach in [1]), which would give the right importance to a node as A in our example.

2.2.2 New proposals

Following Del Corso, Gullí and Romani [18], one objective of the section is a precise policy for providing to any dangling node a link to itself or to itself and its parents, with a given distribution (in order to impose monotonicity, at least in a weak sense). A link to itself models a reload action and hence it could be also used for the other non–dangling nodes.

A second and more relevant objective is to incorporate the transient behavior for differentiating our ranking from the limit solution to the surfing model. This will be done:

- by eliminating the teleportation parameter that induces a confusion between the notion of ranking and the surfing model,
- by introducing a weighting by experience for reinforcing the role of the transient phenomena,
- by using the Cesaro mean for avoiding oscillatory phenomena at the limit,
- by defining a nonlinear model, in the spirit of a dynamical system, for using the computed PageRank at time t in order to update the PageRank at time $t+\Delta t$.

A new policy for dangling nodes

Now we describe a way for implementing weighted self–loops and weighted links to parents.

Let $\mathsf{in}(A)$ be the set of ingoing edges to node A including possibly the node A itself, let $\mathsf{deg}^{-}(A)$ be its cardinality (this number could be theoretically zero), $\mathsf{out}(A)$ be the set of outgoing edges from node A including possibly the node A itself, and let $\mathsf{deg}(A)$ be its cardinality (this number could be theoretically zero and in this case, as already observed, node A is a dangling node). In the following $v[\mathsf{in}(A)]$ is the sum of $v[B]$ for B such that the edge from B to A exists, $v[\mathsf{out}(A)]$ is the sum of $v[B]$ for B such that the edge from A to B exists, v being as usual the personalization vector, and g is a positive

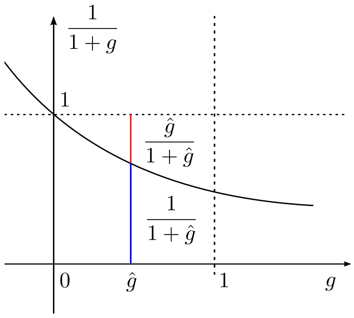


Figure 2.1:

damping parameter. A reasonable choice is g such that $g/(1+g) = 1/10$, i.e., $g = 1/9$ (ref Fig. 2.1).

The resulting policy will be the following (any form 0/0 is set to zero).

Case 1 If $\deg^-(A) = \deg(A) = 0$ then there will be a unique edge from A to A (a loop) with weight of the node equal to 1; in case $v[A] = 0$ the node is simply deleted with its edges.

Case 2 If $\deg(A) = 0$ and $\deg^-(A) > 0$ then there will an edge from A to A with weight

$$v[A]/(v[\mathrm{in}(A)]/\deg^-(A) + v[A])$$

and $\deg^-(A)$ edges from A to B, with B belonging to the set of ingoing nodes of A and with weight

$$v[B]/(v[\mathrm{in}(A)] + \deg^-(A)v[A]);$$

in case $v[\mathrm{in}(A)] + v[A] = 0$ the node is simply deleted with its edges.

Case 3 If $\deg(A) > 0$ and $\deg^-(A) = 0$, then there will an edge from A to A with weight

$$gv[A]/(v[\mathrm{out}(A)]/\deg(A) + gv[A])$$

and $\deg(A)$ edges from A to B, with B belonging to the set of outgoing nodes of A and with weight

$$v[B]/(v[\mathrm{out}(A)] + g\deg(A)v[A]);$$

in case $v[\mathrm{out}(A)] + v[A] = 0$ the node is simply deleted with its edges.

Case 4 Otherwise, there will be an edge from A to A with weight

$$gv[A]\deg^{-}(A)/\{(1+g)(v[\text{in}(A)]+\deg^{-}(A)v[A])\},$$

$\deg(A)$ edges from A to B, with B belonging to the set of outgoing nodes of A and with weight

$$v[B]/\{(1+g)v[\text{out}(A)]\},$$

and $\deg^{-}(A)$ edges from A to B, with B belonging to the set of ingoing nodes of A and with weight

$$gv[B]/\{(1+g)(v[\text{in}(A)]+\deg^{-}(A)v[A])\}.$$

in case $v[\text{in}(A)]+v[\text{out}(A)]+v[A]=0$ the node is simply deleted with its edges.

To have an idea in a concrete but exemplified case, by setting $g=1/9$, i.e., $g/(1+g)=1/10$ and by supposing the personalization vector v uniform, i.e., with all entries equal to $1/n$, the described policy amounts to the following scheme: If $\deg^{-}(A)=\deg(A)=0$ then there will be a unique edge from A to A (a loop) with weight of the node equal to 1; If $\deg(A)=0$ and $\deg^{-}(A)>0$ then there will an edge from A to A with weight $1/2$ and $\deg^{-}(A)$ edges from A to B, with B belonging to the set of ingoing nodes of A and with weight $1/(2\deg^{-}(A))$; If $\deg(A)>0$ and $\deg^{-}(A)=0$, then there will an edge from A to A with weight $1/10$ and $\deg(A)$ edges from A to B, with B belonging to the set of outgoing nodes of A and with weight $9/(10\deg(A))$; Otherwise, there will be an edge from A to A with weight $1/20$, $\deg(A)$ edges from A to B, with B belonging to the set of outgoing nodes of A and with weight $9/(10\deg(A))$, and $\deg^{-}(A)$ edges from A to B, with B belonging to the set of ingoing nodes of A and with weight $1/(20\deg^{-}(A))$.

We observe that is it possible to apply this new policy in the construction of the matrix G of the PageRank model. In this way we get a matrix G that can be reduced into the direct sum of irreducible and primitive blocks (ref Section 3.1) and the Perron Frobenius theory ensures us that there exists always a unique ranking, function of the personalization vector v. It is possible to obtain this solution applying the power method to the matrix

$G(1) = G$. At this point the problem related to the presence of dangling nodes is solved, but transient phenomena are not yet taken into account. A further modification of the PageRank model must be considered.

Introducing transient phenomena

An improved ranking may be computed starting from the uniform vector u with components all equal to $w[0]/n$ (with n being the size of the Web) and adding all the vectors $w[j]P^j e/n$, where $P = G^T$, $G = G(1)$, is the transpose of the Google matrix with parameter $c = 1$, j ranges from 0 to a reasonable number of clicks, $w[j] > 0$ is the jth term of a sequence forming a convergent series. Here the Google matrix is that of the old model with $c = 1$ and classical treatment of dangling nodes: moreover, the present proposal is not limited to choosing a hyperlink at random within a page with uniform distribution; if statistical data are known about actual usage patterns, then that information can be included since any arbitrary distribution u describing the choice of the hyperlink can be considered. Here the speed of the decay of $w[j]$ to zero, as j tends to infinity, can be used for deciding to give more or less importance to the stationary limit distribution (solution to the surfing model) or to the transient behavior. Indeed, if one should choose a page where to put the advertisement of a new product, the user would prefer a page with high transient ranking (transient, i.e., for j moderate e.g. at most 10, 15) because many people will have a chance of looking at it, instead of a page with low transient ranking and high final ranking (final, i.e., as j tends to infinity). In fact no user will wait so much or, if he/she waits on the Web, then he/she will be probably terribly tired and unable to appreciate any commercial suggestion. This can motivate a first concrete proposal of $w[0] = w[1] = \cdots = w[k] = (p-1)/(pk)$, for a reasonably moderate k (e.g. k integer with k in the interval $[7, 20]$), p belonging to $[2, 10]$, and the remaining $w[j]$, $j > k$, such that $w[k+1] + w[k+2] + \cdots = 1/p$. In practice, for j larger of any reasonable number of clicks, dictated e.g. by the "physical resistance" of a generic user, we could set $w[j] = 0$. Furthermore, since the Cesaro sum of the $P^j u$ tends to a stationary distribution (as in the Google model) and this stationary distribution is the limit as the teleportation parameter c tends to 1 of $y(c)$, $y(c)$ being the PageRank, instead of the general condition $w[k+1]+w[k+2]+\cdots=1/p$ we can safely choose $w[k+1]=1/p$, $w[m]=0$ for every m larger than $k+1$ and the classical $y(1)$ instead of $P^{k+1}u$. The choice of $y(1)$ is recommended for stabilizing the computation: indeed the sequence $P^j u$ may fail to converge, while its Cesaro mean converges to the ergodic projector.

A natural problem at this point is: how to manage SPAM pages? An interesting idea used in the previous model is based on a careful choice of the personalization vector v (see below): hence as before, in the previous sum, the uniform vector u is replaced by the personalization vector v.

A second natural problem is the computation of $y(1)$ intended, by definition, as the limit as the teleportation parameter c tends to 1 of $y(c)$ with generic personalization vector v.

In fact from the analysis in [39, 25] we know that $y(c)$ is an analytic function of c on the complex plane, except for a finite number of points different from 1 outside the open unit disk (see Sections 2.3 and 2.4). Therefore $y(1)$ can be approximated, just by continuity, by $y(c)$ with c close to 1 $(0.9, 0.99)$: there is a lot of work by Golub and Greif (using Arnoldi, see [21]), Del Corso, Gullí, Romani (using the linear system representation and preconditioned GMRES, see [18]), Breziski et al. (vector extrapolation based on explicit rational formulae of $y(c)$, see [13, 14, 15]) etc. for making such computations fast. Otherwise the straightforward but effective algorithm in Subsection 2.5.2 can be conveniently employed.

An appropriate choice of the involved parameters, based on the experience, is also possible with special reference to k, p and to the weights $w[j]$. Here is a first embodiment: a visit to the page A will make A more important if it is longer: following this principle the value $w[j]$ could be decided as a monotone function of the average time of a generic user between the click j and the click $j+1$ (see below). While the previous model is trying to rank the importance at the limit (the asymptotic stationary distribution, i.e., the solution to the surfing model), the present approach can be seen as a global ranking, i.e., as a weighted integral over the discrete time (decided by clicks on the Web) of the ranking. Of course, as already informally observed, the weights $w[j]$, like in any weighted quadrature formula, decide where to put the attention for giving the final decision on the global ranking.

Another healthy effect of the integral approach is the stabilization of the involved quantities which prevent from spurious oscillations and this stabilization is typical of any Cesaro like process. Indeed, by considering again the example above, with the old model the ranking of page B and C are oscillating. Depending on starting distribution vector, they exchange the first and the second top positions at every j and the difference between their ranking is not negligible. Of course, the use of teleportation just alleviates the phenomenon, which is eliminated at the limit, but in practice it remains well visible. The averaging implied by the integral approach substantially reduces this fact as any Cesaro like process does: however, it should be noticed that a plain Cesaro approach would again give emphasis only on the limit behavior, since its representing matrix would converge to the spectral projector (see again [33, 40]).

Furthermore, let us give more details on a more accurate proposal for the determination of the sequence $w[j]$, based on experience. Consider for a moment to have the following information on all the visits on the Web for a certain window of observation (one week for instance). Let $\mathrm{surfing[j]}$ with $j \geq 0$ be a nonnegative integer that represents the number of visits to the Web that last at least j clicks. If you are on the Web and you change Web page not clicking, but by writing explicitly the address, then this is counted as a restart, i.e., in the number $\mathrm{surfing}[0]$. Moreover, there exists only a finite number of indices j with nonzero $\mathrm{surfing[j]}$, due to the finiteness of the time interval and due to the physical resistance of the generic user. Now we make a statistic on the lengths t_{j+1}, with $j \geq 0$, of the time intervals between the click number j and the click $j+1$, if the click j is not the last click, or the time intervals between the click number j and the exit, if the click j is the last. Let us denote by $T[j+1]$ the average value of these t_{j+1} based on our observations over all the visits. Then calling $\gamma[j] = \mathrm{surfing[j]} \cdot \mathrm{T[j+1]}$, $j \geq 0$, and $s[h]$ the sum of all $\gamma[j]$ with $j \geq h$, our integral will be

$$y = F(P, v, w) = w[0]v + w[1]Pv + \cdots + w[k]P^k v + w[k+1]y(1) \tag{2.4}$$

with $w[j] = \gamma[j]/s[0]$, $j = 0, \ldots, k$, and $w[k+1] = s[k+1]/s[0]$.

In this way more influence is given to $P^s v$ if the "area" $w[s]$ is maximal: $w[j]$ may be viewed as the area of a rectangle where the length of the basis is the average time between click j and click $j+1$ and the length of the height is equal to the value surfing[j] i.e. the number of visits that last at least j clicks. It is not excluded that the behavior of such a sequence $w[j]$ can be roughly approximated by a Poisson distribution with a given mean.

Along the same line the personalization vector v can be described. It should be nonnegative and with unit l^1 norm (just a matter of scaling). Moreover $v[j]$ should be put at zero if j is recognized as a SPAM page and for the other pages the value $v[j]$ has to be proportional to the sum over the visits to j at the first click of the visit–time.

Of course these parameters have to be estimated, but the leaders of Web–Searching Market (as Google, Microsoft, Yahoo etc.) for sure have access to such information.

We can now apply the new policy for nodes, described in the former Subsection, in the construction of the matrix $P = G^T$ that appears in the formula (2.4). Thanks to the structure of this new P there are no more spurious oscillations in the terms $P^j v$ for j increasing. So instead of $y(1)$ in (2.4) we can consider safely the term $P^{k+1}v$.

$$y = F(P, v, w) = w[0]v + w[1]Pv + \cdots + w[k+1]P^{k+1}v \tag{2.5}$$

with $w[j]$ evaluated as previously proposed.

The ranking coming out of the joining of these two techniques seems to be exempt from the pathologies of the classical PageRank.

Finally, the latter statement suggests to look at the problem in a time dependent and nonlinear way, since the Web evolves in time and the expected values of the various time intervals, i.e. $T[j]$ for $j = 1, 2, \ldots$, also depend on the ranking that we attribute to Web pages. A concrete proposal is the following: if $\hat{y}(t)$ denotes this new definition of the PageRank according to the formula (2.5), then we define the new ranking at $t + \Delta t$ as

$$\hat{y}(t+\Delta t) = F(P(t+\Delta t), z, w(t+\Delta t)), \qquad z = m\hat{y}(t) + (1-m)v(t+\Delta t), \ 0 \le m \le 1,$$

where $P(t + \Delta t)$ is the Web matrix at the time $t + \Delta t$, $w(t + \Delta t)$ is the vector of the weights at the time $t+\Delta t$, and where z is defined as a convex combination of $v(t+\Delta t)$ (the personalization vector defined as before at time $t+\Delta t$) and $\hat{y}(t)$ which carries the information on ranking at the older time t. The nonnegative parameters m and $1-m$ of the convex combination can be interpreted as weights that measure the level of fidelity, which is based on the "past importance".

Further possibilities

In summary two goals are achieved by the new model. The actual efficiency (fast computation) is preserved, since the new computation will involve at most two vectors, which already were computed in the preceding model, and it seems that the old pathologies are removed without introducing new ones. The new ranking method according to the proposal may be called the VisibilityRank or the CommercialRank, since a query–independent measure is given of the "fair value" of any Web page for deciding e.g. the cost of putting an advertisement in that page, as in the determination of the cost of renting a space for advertisement in a given place of a given street, square in a given town etc.

As a final remark on this model part, it is worth mentioning that this model could be of interest not only in Web ranking, but also in political/social sciences e.g. for ranking who/what is influential and who/what is not (as an example one could be interested in answering to the following questions: Bill Clinton's opinion is really influential and at which level? How to rank immaterial forces such as a religious authority vs material forces such as economic/military powers?), in many aspects of marketing, for ranking human resources, for ranking the importance of a paper and/or of a researcher looking in scientific databases, see [4]. Let us think to MathSciNet for Mathematicians where a generic node is any paper in the database and a link from A to B is just a bibliographic reference to paper B in paper A. For evaluating the impact (i.e. the ranking) of a paper the very same model and the same procedure as described before could be applied to the related graph. For evaluating or ranking a researcher (a very hot topic nowadays in several countries) it would be enough to modify the graph where every single node is a researcher and a link from A to B means that the researcher A has written at least one paper referring to at least one paper of the researcher B: the links have to be weighted and the related weights will be proportional to the number of such papers and will be properly normalized according to the number of authors in the referring papers of A and in the referred papers of B. The algorithm will be again the same and again the same idea would work for ranking researcher groups or Institutions such as Departments, Faculties, Universities (see e.g. the hierarchical approach in European Patent 1, 653, 380 A1). In addition it is worth stressing that the described procedures for defining the graph and for computing the ranking are unchanged in any Scientific homogeneous community.

Of course, for modeling in a convincing way such complex phenomena, it would be recommended to enrich the structure of the graph by adding to nodes and/or to edges more information (meta–graph? ...). However, the essential basic idea for defining and computing the ranking has to remain virtually the same.

2.3 The general parametric Google matrix

We begin with a summary of the properties of a row–stochastic matrix that are relevant to our analysis of the general parametric Google matrix.

2.3.1 Lemma. [2, 25] *Let A be a row–stochastic matrix. Then*

(i) $\lambda = 1$ is an eigenvalue of A associated with the right eigenvector $x = e$.

(ii) Every entry of A is in the real interval $[0,1]$.

(iii) For each $k = 1,2,\ldots$, A^k is row–stochastic, so its entries remain bounded as $k \to \infty$.

(iv) Every eigenvalue of A has modulus at most 1.

(v) Every eigenvalue of A that has modulus 1 is semisimple.

(vi) If the eigenvalue 1 has multiplicity m, then the Jordan canonical form of A is

$$I_m \oplus J_{n_1}(\nu_1) \oplus \cdots \oplus J_{n_k}(\nu_k),$$

in which each $\nu_j \neq 1$, each $|\nu_j| \leq 1$, and $n_j = 1$ if $|\nu_j| = 1$.

(vii) If 1 *is a simple eigenvalue of* A*, then there is a unique vector* y *with nonnegative entries such that* $y^T A = y^T$ *and* $y^T e = 1$*.*

Since the basic Google matrix G has all the properties stated in the preceding lemma, and since these properties are special cases of the key hypotheses in our analysis in the preceding Chapter, specialization of our general results permits us to identify several pleasant and useful properties of the *general parametric Google matrix* $G(c) = cG + (1-c)xv^*$ with complex c and v. In fact the following Theorem is an interpretation of Theorems 1.4.1 and 1.4.2 when A is the Google matrix and hence $\lambda = 1$.

2.3.2 Theorem. *Let* G *be an* n*–by–*n *row stochastic matrix, and let its eigenvalue* $\lambda = 1$ *(necessarily semisimple) have multiplicity* $m \geq 1$*. If* $m = 1$*, let* y *be the unique vector with nonnegative entries such that* $y^T G = y^T$ *and* $y^T e = 1$*. If* $m > 1$*, let the* m *columns of* X *be any linearly independent set of right* 1*–eigenvectors of* G*, and let* Y *be the matrix defined in Theorem 1.4.1(ii); its columns are an independent set of left* 1*–eigenvectors of* G*. Let* v *be a given complex vector such that* $v^* e = 1$*, let* c *be a complex number, and let* $G(c) = cG + (1-c)ev^*$*. Let* $1, \lambda_2, \ldots, \lambda_n$ *be the eigenvalues of* G*, let* $\mu_1, \ldots, \mu_d$ *be the nonzero eigenvalues of* G *that are different from* 1*, let*

$$\varepsilon = \min\{|1-\mu_1^{-1}|, \ldots, |1-\mu_d^{-1}|\},$$

and let

$$I_m \oplus J_{n_1}(\nu_1) \oplus \cdots \oplus J_{n_k}(\nu_k), \text{ each } \nu_j \neq 1 \tag{2.6}$$

be the Jordan canonical form of G*, with* $\{\mu_1, \ldots, \mu_d\} \subseteq \{\nu_1, \ldots, \nu_k\} \subseteq \{\lambda_2, \ldots, \lambda_n\}$*. Then*

(i) The eigenvalues of $G(c)$ *are* $1, c\lambda_2, \ldots, c\lambda_n$*, and* $|c\lambda_j| \leq |c|$ *for each* $j = 2, \ldots, n$*.*

(ii) In the Jordan canonical form (2.6), $n_j = 1$ *for each* j *such that* $|\nu_j| = 1$*.*

(iii) If $0 < |c| < 1$ *(or, more generally, if* $c \neq 0$ *and* $1 \neq c\mu_j$ *for each* $j = 1, \ldots, d$*), then the Jordan canonical form of* $G(c)$ *is*

$$[1] \oplus cI_{m-1} \oplus J_{n_1}(c\nu_1) \oplus \cdots \oplus J_{n_k}(c\nu_k)$$

if $m > 1$*; it is*

$$[1] \oplus J_{n_1}(c\nu_1) \oplus \cdots \oplus J_{n_k}(c\nu_k)$$

if $m = 1$.

(iv) Suppose either that $|c| < 1$ *or that* $0 < |1-c| < \varepsilon$. *Then* 1 *is a simple eigenvalue of* $G(c)$.

(v) Suppose either that $|c| < 1$ *or that* $0 < |1-c| < \varepsilon$. *If* $m > 1$, *the unique left* 1*–eigenvector* $y(c)$ *of* $G(c)$ *such that* $y(c)^* e = 1$ *is defined by*

$$y(c)^* = v^* XY^* + (1-c)v^* S_2 (I_{n-m} - cE)^{-1} Z_2^*, \tag{2.7}$$

and

$$\lim_{c \to 1} y(c) = YX^* v. \tag{2.8}$$

The matrices S_2, E, *and* Z_2 *are defined in Theorem 1.4.1(ii);* 1 *is not an eigenvalue of* E. *The matrix*

$$YX^* = I - (I - G^T)(I - G^T)^D \tag{2.9}$$

is a real projection with nonnegative entries.

(vi) Suppose either that $|c| < 1$ *or that* $0 < |1-c| < \varepsilon$. *If* $m = 1$, *the unique left* 1*–eigenvector* $y(c)$ *of* $G(c)$ *such that* $y(c)^* e = 1$ *is defined by*

$$y(c)^* = y^* + (1-c)v^* S_1 (I_{n-1} - cB)^{-1} Z_1^*, \tag{2.10}$$

and

$$\lim_{c \to 1} y(c) = y. \tag{2.11}$$

The matrices S_1, Z_1, *and* B *are defined in Theorem 1.4.1(i);* 1 *is not an eigenvalue of* B.

(vii) The vector function $y(c)$ *defined by (2.7) if* $m > 1$, *and by (2.10) if* $m = 1$, *is analytic in the unit disk* $\{c : |c| < 1\}$ *and is represented there by the Maclaurin series*

$$y(c)^* = v^* \left(I + \sum_{k=1}^{\infty} \left((G - I) G^{k-1} \right) c^k \right). \tag{2.12}$$

(viii) Let $\gamma = c-1$. The vector function $y(c)$ defined by (2.7) if $m>1$, and by (2.10) if $m=1$, is analytic in the disk $\{c : |1-c| < \varepsilon\}$ and is represented there by the power series

$$y(c)^* = y(\gamma+1)^* = v^* \left(XY^* - \sum_{k=1}^{\infty} \left(((I-G)^D)^k G^{k-1} \right) \gamma^k \right). \tag{2.13}$$

In particular, the first derivative at $c=1$ is

$$y'(1)^* = v^*(G-I)^D. \tag{2.14}$$

(ix) Let K be a given compact complex set that does not contain any of the points $\mu_1^{-1}, \ldots, \mu_d^{-1}$. Define $y(c)$ on K by (2.7) if $m>1$ and by (2.10) if $m=1$. Then $\|y(c)\|_1 \geq 1$ for all $c \in K$ and there is a positive constant M such that

$$\frac{\|y(\tilde{c}) - y(c)\|_1}{\|y(c)\|_1} \leq \|y(\tilde{c}) - y(c)\|_1 \leq M|\tilde{c} - c| \text{ for all } \tilde{c}, c \in K.$$

The assertions in (vii) and (viii) of the preceding Theorem follow from (1.24), (1.26), and (1.27). The assertion (ix) follows from Theorem 1.4.1(iii) and the observation that $1 = |y(c)^* e| \leq \|y(c)^*\|_1$.

We emphasize that the representations (2.7) and (2.10) for the unique normalized left 1–eigenvector of $G(c)$ are valid not only for all real $c \in (0,1)$, but also for all complex c in the open unit disk, as well as for all c in a punctured open neighborhood of the point 1 in the complex plane. The limits (2.8) and (2.11) are to be understood as limits of functions of a complex variable; the existence of these limits ensures that they may be computed via any sequence of values of c that tends to 1.

The preceding comments have an important consequence. Suppose the vector v has positive real entries and satisfies $v^T e = 1$. Then for all real c such that $0 < c < 1$, $G(c)$ has positive entries. The Perron–Frobenius Theorem ensures that each such $G(c)$ has a unique left 1–eigenvector $y(c)$ that has positive entries and satisfies $y(c)^T e = 1$. Theorem 2.3.2 ensures that $\lim_{c \to 1} y(c) = \tilde{y}$ exists, so if we take this limit with $c \in (0,1)$ we know that $\tilde{y}$ has real nonnegative entries. However, we can also take this limit with c tending to 1 along some non–real path in the complex plane. Regardless of the path taken, and even though $y(c)$ can be non–real on that path, nevertheless the limit obtained is always the nonnegative vector $\tilde{y}$ (this further degree of freedom is exploited in the algorithm presented in Section 2.5).

We can draw one more conclusion from the preceding discussion, which is the last statement in Theorem 2.3.2(v). For each given nonnegative vector v, we have argued that the vector

$$\widetilde{y} = \lim_{c\to 1} y(c) = YX^* v$$

has nonnegative entries. But a matrix N has the property that the entries of Nv are nonnegative whenever the entries of v are nonnegative if and only if all the entries of N are nonnegative. Thus, the projection

$$YX^* = N = [\eta_1 \ \cdots \ \eta_n] = I - (I - G^T)(I - G^T)^D \tag{2.15}$$

is both real and nonnegative. Its columns $\eta_1, \ldots, \eta_n$ are a uniquely determined set of nonnegative left 1–eigenvectors of G such that, for any given nonnegative probability vector v, $\lim_{c\to 1} y(c) = v_1\eta_1 + \cdots + v_n\eta_n$ is a convex combination of them. This matrix is a nonnegative projector coinciding with the Cesaro mean $\lim_{r\to\infty} \frac{1}{r+1} \sum_{j=0}^{r} G^j(1)$: the result is due to Lasserre, who calls N the *ergodic projector* in a context of probability theory [33].

Of course the nonnegativity of N could be obtained elementarily also by using Markov chains arguments.

2.4 Comparison with an explicit prior expression of Google Jordan form

We now consider a result from [39] and we ask ourselves how one can obtain, extend, and interpret them, by employing our findings in the previous sections and by allowing the parameter c in the complex field.

Indeed, we study the Jordan form general case, in which G is not necessarily diagonalizable, where the decomposition $G = SJS^{-1}$,

$$J = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & \blacklozenge & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \lambda_{n-1} & \blacklozenge \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{bmatrix} \tag{2.16}$$

is the Jordan Canonical Form of G and $\blacklozenge$ denotes a value that can be either 0 or 1.

2.4.1 Theorem. *Let G be a row stochastic matrix of size n, let $c \in (0,1)$, and suppose that v is a nonnegative n–vector whose entries add to 1. Consider the matrix $G(c) = cG + (1-c)ev^T$ and let $G = SJ(1)S^{-1}$, $S = [e\ x_2\ \cdots\ x_n]$, $[S^{-1}]^T = [y\ \ y_2\ \cdots\ y_n]$, and*

$$J(c) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & c\lambda_2 & c\cdot\blacklozenge & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & c\lambda_{n-1} & c\cdot\blacklozenge \\ 0 & \cdots & \cdots & 0 & c\lambda_n \end{bmatrix},$$

$$J(c) = D^{-1} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & c\lambda_2 & \blacklozenge & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & c\lambda_{n-1} & \blacklozenge \\ 0 & \cdots & \cdots & 0 & c\lambda_n \end{bmatrix} D \tag{2.17}$$

in which $D = \operatorname{diag}(1, c, \ldots, c^{n-1})$ and $\blacklozenge$ denotes a value that can be 0 or 1. Then

$$G(c) = ZJ(c)Z^{-1},$$

in which

$$Z = SR^{-1},$$

$$R = I_n + e_1 w^T, \quad w^T = [0 \; w[2] \; \cdots \; w[n]],$$
$$w[2] = (1-c)v^T x_2/(1-c\lambda_2), \tag{2.18}$$
$$w[j] = [(1-c)v^T x_j + [J(c)]_{j-1,j} w[j-1]]/(1-c\lambda_j), \; j = 3,\ldots,n. \tag{2.19}$$

In particular

$$y(c) = y + \sum_{j=2}^{n} w[j] y_j \tag{2.20}$$

where $y = y(1)$ *if the eigenvalue* 1 *of* $G = G(1)$ *is simple and where the quantities* $w[j]$ *are expressed as in (2.18)–(2.19). Conversely,* y *is one of the basic PageRank vectors when the eigenvalue* 1 *of* $G(1)$ *is semisimple but not simple.*

Notice that in the original paper [39], there is a typo since D and D^{-1} are exchanged in (2.17): we thank Gang Wu and Yimin Wei for pointing this out to our attention, see [42].

2.4.1 Matching old and new representations

Here we make a critical analysis of the above results in the light of the conclusions in Section 2.3. From Lemma 2.3.1(vi) and Theorem 2.3.2 we know that the eigenvalue 1 in the matrix $G = G(1)$ is semisimple with multiplicity m. Therefore $[J(c)]_{j-1,j} = 0$ and $1 - c\lambda_j = 1 - c$, $j = 2,\ldots,m$. Hence, as already acknowledged in [39][Section 3], the coefficient $w[j]$, $j = 2,\ldots,m$, is equal to $v^T x_j = x_j^T v$ and then

$$y(c) = y + \sum_{j=2}^{m} y_j (x_j^T v) + \sum_{j=m+1}^{n} w[j] y_j.$$

Therefore the Cesaro averaging projector N already discussed in the previous sections has the form $N = YX^* = [y \; y_2 \; \cdots \; y_m][e \; x_2 \; \cdots \; x_m]^T$ and hence $y(c) = Nv + \sum_{j=m+1}^{n} w[j] y_j$. Moreover the eigenvalue λ_j, $j \geq m+1$, is different from 1, is in modulus bounded by 1, and if unimodular then it is semisimple. Consequently, by (2.18)–(2.19), we obtain $\lim_{c\to 1} w[j] = 0$, $j = m+1,\ldots,n$, so that

$$\lim_{c\to 1} y(c) = Nv$$

which agrees with (1.28), (2.8), and (2.9): moreover, by the general reasoning at the end of Section 2.3, we deduce that N is entry–wise nonnegative.

Now, by taking into account the notations in (2.6) considered in Theorem 2.3.2, and by looking carefully at the expression of coefficients $w[j]$, $j = m+1,\ldots,n$, in (2.18)–(2.19), we can rewrite the vector $y(c)$ as

$$\begin{aligned} y(c) &= Nv + \sum_{j=m+1}^{n} w[j] y_j = \\ &= Nv + (1-c)\sum_{j=1}^{k}\sum_{s=1}^{n_j}\sum_{t=1}^{n_j+1-s} c^{t-1}(1-c\nu_j)^{-t}(x_{j,s}^T v)\ y_{j,t}, \end{aligned} \tag{2.21}$$

where the vectors x_j, y_j, $j = m+1,\ldots,n$, in the former representation, have been reorganized according to the Jordan structure as $x_{j,s}, y_{j,s}$, $j = 1,\ldots,k$, $s = 1,\ldots,n_j$ (ref Theorem 2.3.2). If we compare the latter equation with the Toeplitz matrices (Toeplitz, i.e., constant along diagonals, see e.g. [10]) of size n_j

$$J_{n_j}(\nu_j) = \begin{bmatrix} \nu_j & 1 & 0 & \cdots & 0 \\ 0 & \nu_j & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \nu_j & 1 \\ 0 & \cdots & \cdots & 0 & \nu_j \end{bmatrix},$$

$$\begin{aligned} T_{n_j}(c) &= \begin{bmatrix} 1-c\nu_j & -c & 0 & \cdots & 0 \\ 0 & 1-c\nu_j & -c & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 1-c\nu_j & -c \\ 0 & \cdots & \cdots & 0 & 1-c\nu_j \end{bmatrix}^{-1} = \\ &= \frac{1}{1-c\nu_j}\begin{bmatrix} 1 & \frac{c}{1-c\nu_j} & \frac{c^2}{(1-c\nu_j)^2} & \cdots & \frac{c^{n_j-1}}{(1-c\nu_j)^{n_j-1}} \\ 0 & 1 & \frac{c}{1-c\nu_j} & \cdots & \frac{c^{n_j-2}}{(1-c\nu_j)^{n_j-2}} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 1 & \frac{c}{1-c\nu_j} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}, \end{aligned}$$

we observe $T_{n_j}(c) = (I_{n_j} - cJ_{n_j}(\nu_j))^{-1}$ and therefore

$$\sum_{s=1}^{n_j} \sum_{t=1}^{n_j+1-s} c^{t-1}(1-c\nu_j)^{-t}(x_{j,s}^T v)\; y_{j,t} = [y_{j,1} \;\cdots\; y_{j,n_j}] T_{n_j}^T(c) [x_{j,1} \;\cdots\; x_{j,n_j}]^T v.$$

Hence, taking into account (2.21), we can write

$$\begin{aligned} y(c) &= Nv + (1-c)\sum_{j=1}^{k} [y_{j,1} \;\cdots\; y_{j,n_j}] \cdot \\ &\quad \cdot (I_{n_j} - cJ_{n_j}^T(\nu_j))^{-1}[x_{j,1} \;\cdots\; x_{j,n_j}]^T v \\ &= (1-c)(I - cG^T)^{-1} v \end{aligned} \tag{2.22}$$

which coincides with the general representation (1.15), where $X = [e \; x_2 \;\cdots\; x_m]$, $Y = [y \; y_2 \;\cdots\; y_m]$, $N = YX^T$, $E = J_{n_1}(\nu_1) \oplus \cdots \oplus J_{n_k}(\nu_k)$, as in the expression (2.6), and $S_2 = [X_1 \;\cdots\; X_k]$, $Z_2 = [Y_1 \;\cdots\; Y_k]$, $X_j = [x_{j,1} \;\cdots\; x_{j,n_j}]$, $Y_j = [y_{j,1} \;\cdots\; y_{j,n_j}]$, $j = 1, \ldots, k$.

2.4.2 Eigenvector structure of $G(c)$, discontinuity points in its Jordan form

When writing the Jordan form in Theorem 2.4.1, the matrix D is chosen as

$$\mathrm{diag}(1, c, \ldots, c^{n-1}).$$

However, that matrix is not unique: for instance the matrix

$$\hat{D} = I_m \oplus \mathrm{diag}(1, c, \ldots, c^{n-m-1})$$

is also a feasible choice, since the Jordan structure of $G(c)$ is equally obtained as $DJ(c)D^{-1} = \hat{D}J(c)\hat{D}^{-1}$. Indeed, following the Jordan blocks structure in (1.15), we can define a new optimal diagonal matrix $\widetilde{D}$ of minimal conditioning with the constraint that $DJ(c)D^{-1} = \widetilde{D}J(c)\widetilde{D}^{-1}$. This optimal matrix takes the form

$$\widetilde{D} = I_m \oplus \mathrm{diag}(1, c, \ldots, c^{n_1-1}) \oplus \mathrm{diag}(1, c, \ldots, c^{n_2-1}) \oplus \cdots \oplus \mathrm{diag}(1, c, \ldots, c^{n_k-1}).$$

Therefore, switching from $G = G(1)$ to $G(c)$, while the eigenvalues change in a smooth way since $1 \to 1$ with the same multiplicity m, $\nu_j \to c\nu_j$, $j = 1, \ldots, k$, the left and right vectors change as follows

$$\begin{aligned}
&x_{j,t} \to c^{1-t}\left[x_{j,t} - (1-c)\tfrac{c^t}{1-c\nu_j}e\right], && y_{j,t} \to c^{t-1}y_{j,t}, && t = 1,\ldots,n_j,\\
&x_t \to x_t - e, && y_t \to y_t, && t = 2,\ldots,m,\\
&x_1 \equiv e \to e, && y_1 \equiv y \to y(1) = Nv.
\end{aligned}$$

Therefore, in the given representation and under the assumption of non-diagonalizable G, the Jordan canonical form has a discontinuity at $c = 0$, while it behaves smoothly at $c = 1$. In fact, $\lim_{c\to 0} G(c) = ev^T$ is not normal in general but it is diagonalizable, while $G(c)$ with $c \neq 0$ is not diagonalizable in general: in fact $G(c)$ has the same Jordan pattern as $G(1)$ for $c \neq 0$ while it is diagonalizable for $c = 0$. Hence, as emphasized in the previous displayed equations, it is clear that the discontinuity/degeneracy is located in the left and right vectors associated with nontrivial Jordan blocks. Consequently the matrix $G(c)$ is continuous at $c = 0$, but it is not so for its Jordan representation. On the other hand the other discontinuities at $c = \nu_j^{-1}$, for every $j = 1,\ldots,k$, are essential not only in the representations, but also in the matrix $G(c)$, and at the point $c = 1$ every involved quantity is analytic.

Finally it should be noted the following "surprising" fact: not only nothing bad happens at $c = 1$, but indeed nothing bad happens for $c > 1$ (at least, a little bit bigger than 1) and this is not seen by the power series representations of $y(c)$ described in the literature, which diverge for $c > 1$ (see [9]).

2.4.3 Condition number of $y(c)$: general derivation

Given its relevance for numerical stability, we consider in some detail the conditioning of $y(c)$ in several norms and especially in the more natural l^1 norm. More precisely, we are interested in estimating

$$\kappa(y(c),\delta) = \frac{\|y(\tilde{c}) - y(c)\|}{\|y(c)\|},$$

with $\tilde{c} = c(1+\delta)$, δ complex parameter of small modulus, K compact set in the complex field nonintersecting $\{\mu_j^{-1} : \ j = 1,\ldots,d\}$, and $c,\tilde{c} \in K$. Since $y(c)$ is analytic in its domain, it is clear that

$$\kappa(y(c),\delta) = \kappa_c \frac{|c\delta|}{\|y(c)\|}(1 + O(\delta))$$

with $\kappa_c = \|y'(c)\|$. Our next task is the differentiation of $y(c)$ in the light of (2.22), and especially its norm evaluation. We have

$$\begin{aligned} y'(c) &= -\sum_{j=1}^{k} [y_{j,1} \ \cdots \ y_{j,n_j}](I_{n_j} - cJ_{n_j}^T(\nu_j))^{-1}[x_{j,1} \ \cdots \ x_{j,n_j}]^T v + \\ &\quad +(1-c)\sum_{j=1}^{k} [y_{j,1} \ \cdots \ y_{j,n_j}](I_{n_j} - cJ_{n_j}^T(\nu_j))^{-2} J_{n_j}^T(\nu_j)[x_{j,1} \ \cdots \ x_{j,n_j}]^T v, \end{aligned} \tag{2.23}$$

which of course agrees with the differentiation of (2.21), after observing that

$$(I_{n_j} - cJ_{n_j}(\nu_j))^{-2} J_{n_j}(\nu_j) = \frac{1}{1-c\nu_j} \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & t_{n_j-1} \\ & t_0 & t_1 & \cdots & t_{n_j-2} \\ & & \ddots & \ddots & \vdots \\ & 0 & & t_0 & t_1 \\ & & & & t_0 \end{bmatrix}$$

with $t_s = \frac{sc^{s-1}}{(1-c\nu_j)^s} + \frac{(s+1)\nu_j c^s}{(1-c\nu_j)^{s+1}}$, $s = 0, \ldots, n_j - 1$. In fact, upper triangular Toeplitz matrices form a commutative algebra and the generic coefficient on the diagonal in the result is a simple convolution of the coefficients of the factors. Therefore, putting the two terms of (2.23) together, we find

$$\begin{aligned} y'(c) &= \sum_{j=1}^{k} [y_{j,1} \ \cdots \ y_{j,n_j}] \widetilde{T}_{n_j}^T(c)[x_{j,1} \ \cdots \ x_{j,n_j}]^T v \\ &= \sum_{j=1}^{k} Y_j \widetilde{T}_{n_j}^T(c) X_j^T v \\ &= Z_2 \left[\oplus_{j=1}^{k} \widetilde{T}_{n_j}^T(c)\right] S_2^T v, \end{aligned} \tag{2.24}$$

with

$$\widetilde{T}_{n_j}(c) = \frac{1}{1-c\nu_j} \begin{bmatrix} \widetilde{t}_0 & \widetilde{t}_1 & \widetilde{t}_2 & \cdots & \widetilde{t}_{n_j-1} \\ & \widetilde{t}_0 & \widetilde{t}_1 & \cdots & \widetilde{t}_{n_j-2} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \widetilde{t}_0 & \widetilde{t}_1 \\ & & & & \widetilde{t}_0 \end{bmatrix},$$

$\widetilde{t}_s = -\frac{c^s}{(1-c\nu_j)^s} + (1-c)\left[\frac{sc^{s-1}}{(1-c\nu_j)^s} + \frac{(s+1)\nu_j c^s}{(1-c\nu_j)^{s+1}}\right]$, $s = 0, \ldots, n_j - 1$, and with $S_2 = [X_1 \ \cdots \ X_k]$, $Z_2 = [Y_1 \ \cdots \ Y_k]$.

Therefore looking at the dependence with respect to the parameter c we find that κ_c grows generically, in a neighborhood of 1, μ_j^{-1}, $j=1,\ldots,d$, as

$$\max_{j=1,\ldots,k}\left|\frac{\widetilde{t}_{n_j-1}}{1-c\nu_j}\right|, \quad \frac{\widetilde{t}_{n_j-1}}{1-c\nu_j} = \\ \left[z_1(1-c\nu_j)^{-n_j}+z_2(1-c\nu_j)^{-n_j-1}(1-c)\right]c^{n_j-2}, \qquad (2.25)$$

$z_1=(n_j-1)(1-c)-c$, $z_2=c\nu_j n_j$, which agrees with the estimate in the introduction (see (2.39)–(2.40)). More precisely, for almost every v nonnegative and with unit l^1 norm, there exists a positive constant $\theta=\theta(S,v)$, independent of c, such that

$$\kappa_c \geq \theta \max_{j=1,\ldots,k}\left|z_1(1-c\nu_j)^{-n_j}+z_2(1-c\nu_j)^{-n_j-1}(1-c)\right||c|^{n_j-2}. \qquad (2.26)$$

In fact by elementary measure theory argument, the set of all possible v such that $x_{j,s}^T v=0$ for at least one index $j=1,\ldots,k$ and one index $s=1,\ldots,n_j$ has zero Lebesgue measure. On the other hand, taking into account (2.24), a direct majorization of the quantity κ_c leads to

$$\kappa_c \leq \sum_{j=1}^{k}\|Y_j\|\|\widetilde{T}_{n_j}^T(c)\|\|X_j^T v\|$$

and to the more appealing

$$\kappa_c \leq \|Z_2\|\cdot\|\oplus_{j=1}^{k}\widetilde{T}_{n_j}^T(c)\|\cdot\|S_2^T v\|. \qquad (2.27)$$

If we take reasonable norms as the l^p norms with $p\in[1,\infty]$, then by recalling $G=G(1)=SJS^{-1}$ and since S_2, Z_2 are submatrices of S, S^{-1} respectively, the bound in (2.27) directly implies the following

$$\kappa_c \leq \kappa(S)\max_{j=1,\ldots,k}\left\|\widetilde{T}_{n_j}^T(c)\right\|\|v\|. \qquad (2.28)$$

In other words, the first part which does not depend on c, tells us that the conditioning of $y(c)$ can be associated with the (lack of) orthogonality of the left and right vectors in the Jordan form of G not associated with the eigenvalue 1, while the second part carries the information on the parameter c. Notice that the generic, lower, and upper bounds in (2.25)–(2.28) are all well defined also at $c=1$, and indeed the latter improves the estimates known in the literature, where, for $0 \le c < 1$, the amplification factor upper–bound grows as $(1-c)^{-1}$ and blows up at $c = 1$: see [30] and references there reported; however, it has to be pointed out that these estimates in [30] are more general since they are based on an arbitrary perturbation $\widetilde{G}$ of $G = G(1)$ subject to the only constraint that $\tilde{G}$ is irreducible and stochastic. Furthermore, for c in the unit disk and far away from $c = 1$, the obtained amplification factor is simpler and more useful, i.e., $(1-|c|)^{-1} + |1-c|(1-|c|)^{-2}$ which reduces to $2(1-c)^{-1}$ for $0 \le c < 1$: therefore our more detailed analysis is of interest essentially in the vicinity of critical points $c = \mu_j^{-1}$, $j = 1, \ldots, d$, $c = 1$, all outside or on the frontier of the unity disk.

2.4.4 Condition number of $y(c)$: norm analysis of $\widetilde{T}_{n_j}^T(c)$

A critical analysis of (2.28) shows that the quantities $\kappa(S)$ and $\|v\|$ are fixed data of the problem ($G = G(1)$ and v); in particular, since $\|v\|_1 = 1$ and $\|\cdot\|_p \le \|\cdot\|_1$, $p \in [1, \infty]$, we uniformly have $\|v\|_p \le 1$. Hence we should focus our attention on $\left\|\widetilde{T}_{n_j}^T(c)\right\|$, $j = 1, \ldots, k$.

For instance, by considering the l^1 and the l^∞ norms, we have

$$\left\|\widetilde{T}_{n_j}^T(c)\right\|_1 = \left\|\widetilde{T}_{n_j}^T(c)\right\|_\infty = \sum_{s=0}^{n_j-1} \left|\frac{\widetilde{t}_s}{1-c\nu_j}\right|,$$

which grows as $\frac{\widetilde{t}_{n_j-1}}{1-c\nu_j}$, for c in a neighborhood of μ_j^{-1}. However, $|\mu_j^{-1}| \ge 1$ while, especially for computational purposes, we are more interested in the behavior of the conditioning for c of modulus at most 1.

In such a case, independently of the chosen norm among l^1, l^2, l^∞, we observe the following: for c such that $|1-c\nu_j| < |c|$, the conditioning of $\widetilde{T}_{n_j}^T(c)$ grows exponentially with the size n_j of the Jordan blocks; of course, also for Jordan blocks of moderate size, the conditioning can become very high. For $|1-c\nu_j| = |c|$, it is clear that the conditioning grows as n_j^2 which can become large only for quite high–dimensional Jordan blocks. For $|1-c\nu_j| > |c|$ the situation is very interesting because, irrespectively of the size n_j the conditioning is bounded. Indeed, by looking at the induced l^2 (the spectral norm), classical results on Toeplitz operators (see the Szegö distribution result in the classical Böttcher, Silbermann book [10]) tell us that there exists a proper function $g_{j,c}(t)$ defined on $[0,2\pi)$

$$\left\|\widetilde{T}_{n_j}^T(c)\right\|_2 \leq \|g_{j,c}(t)\|_\infty, \qquad \lim_{n_j\to\infty}\left\|\widetilde{T}_{n_j}^T(c)\right\|_2 = \|g_{j,c}(t)\|_\infty.$$

That function $g_{j,c}(t)$ called symbol is obtained through the coefficients of $\widetilde{T}_{n_j}(c)$ in the sense that these coefficients are Fourier coefficients of $g_{j,c}(t)$. In our specific setting a straightforward computation shows that the symbol $g_{j,c}(t)$ is

$$\frac{\partial}{\partial c}(1-c)(1-c[\nu_j+\exp(-\mathbf{i}t)])^{-1} = (\nu_j-1+\exp(-\mathbf{i}t))(1-c[\nu_j+\exp(-\mathbf{i}t)])^{-2}, \quad \mathbf{i}^2=-1.$$

Therefore the quantity

$$\max_{t\in[0,2\pi)}\left|(\nu_j-1+\exp(-\mathbf{i}t))(1-c[\nu_j+\exp(-\mathbf{i}t)])^{-2}\right|$$

represents a tight measure, irrespectively of n_j, of the contribution of $\widetilde{T}_{n_j}^T(c)$ to the conditioning of $y(c)$ in l^2 norm. In this context a tight measure of the l^1 norm would have been more desirable, since the l^1 norm represents the most natural choice for the problem at hand.

2.4.5 Condition number of $y(c)$: extremal examples

Here we are interested in showing two extremal examples taken from very structured Web graphs. The Web graph is the one produced by a unique huge loop: page i links only to page $i+1$, $i=1,\dots,n-1$, page n links only to page 1; since the set of dangling nodes is empty the matrix $G=G(1)$ is a special cyclic permutation matrix which generates the algebra of circulants. Circulant matrices are normal and diagonalized by the discrete Fourier transform so that in the Jordan form we have $x_j=f_j$, $y_j=\bar{f}_j$ with

$$f_j = \frac{1}{\sqrt{n}}\left(\exp\left(-\frac{\mathbf{i}2\pi jk}{n}\right)\right)_{k=0}^{n-1}, \qquad j=0,\dots,n-1.$$

The eigenvalues of $G=G(1)$, accordingly to the same ordering of the Fourier eigenvectors, are $\omega_j=\exp\left(-\frac{\mathbf{i}2\pi j}{n}\right)$, $j=0,\dots,n-1$ (the n roots of unity). We notice that e, the used vector of all ones, coincides with $\sqrt{n}f_0$. Therefore if the rank–one correction is chosen with $v=e/n=f_0/\sqrt{n}$, then ev^T is also a circulant matrix. In this specific example the computed vector $y(c)$ coincides with v independently of c and therefore $k_c = 0$. Therefore for this given graph, the chosen vector v lies in the zero measure set excluded when deriving (2.26). In fact, for this graph and for this vector v we have that the vectors x_j, $j = m+1, \dots, n$, $m = 1$, are all orthogonal to v and then the whole expression in (2.23) trivially vanishes.

More delicate is to try to satisfy (2.28) with equality. For important examples the estimate is not tight, but it is not too bad at least in a neighborhood of $c = 1$. Take the above graph, consider $v = e_1$ with $c = 1$. In such a case the estimate (2.28) of κ_1 gives

$$|1-\omega_1|^{-1} = \left[|1-\cos(2\pi(n))|^2 + \sin^2(2\pi/n)\right]^{-1/2} \sim n/2\pi.$$

A direct computation of $y'(c)$ at $c = 1$ gives the expression

$$y'(1) = -\sum_{j=1}^{n-1} \bar{f}_j (1-\omega_j)^{-1} (f_j^T e_1)$$

Since $\bar{f}_j$, $j = 1, \dots, n-1$, are orthonormal and since $|f_j^T e_1| = 1/\sqrt{n}$ it easily follows that

$$\|y'(1)\|_2 = \sqrt{\sum_{j=1}^{n-1} (\sqrt{n}|1-\omega_j|)^{-2}} \sim \sqrt{n} \sqrt{\sum_{j=1}^{n-1} (2\pi j)^{-2}}$$

so that the real l^2 norm of $y'(c)$ differs, asymptotically, from the bound (2.28) by a factor $\sqrt{n}$.

2.5 Computational suggestions

The spectral structure of $G(c)$ was first comprehended in the context of sophisticated results about Markov chains, which required that $c \in [0,1)$ and $v \geq 0$. We now know that the spectral (indeed, the Jordan, Schur etc.) structure of $G(c)$ follows from basic matrix analytic facts that permit both c and v to be complex. This new freedom in the Google perturbation is exploited to compute the PageRank more efficiently, especially when c is close to 1 or even equal to 1.

The algorithms that we propose have to be regarded as a preliminary step that, in our opinion, merits further research.

We choose p small integer number (let us say $p=10$) and we compute $y(c_j)$, $j=0,\ldots,p-1$, at equally–spaced points c_j, $j=0,\ldots,p-1$, on the complex circle of radius (let us say) 0.5 or 0.25. The computations are extremely fast since the standard power method at the kth iteration converges with a relative reduction error of at least $|c|^k$ (see [22, Chapter 7, p. 330]), which is independent of the huge size of the problem; indeed, the nature of our data permits us to use a vector–valued DFT procedure, whose numerical stability is excellent. We employ these p vectors as a starting point for a specific extrapolation algorithm at $c=0.85$ or $c=0.99$, whose details are given in [15, 13]. The idea is to use the expansion of $y(c)$ around $c=1$ as in (2.13) with $\gamma=c+1$ or as in (2.7) or as in (2.18)–(2.20), and to employ linear combinations in order to cancel out certain terms in the remainder $y(c)-\widetilde{y}$, $\tilde{y}=y(1)=Nv$, and to increase the accuracy; see [12, Chapter 4] for details. The vector $\hat{y}(c)$, computed by extrapolation, will be corrupted by errors of approximation and due to roundoff: therefore, since we know in advance that $y(c)$ has to be nonnegative and normalized, we set to zero the real part whenever negative and the imaginary part, and we normalize the resulting nonnegative vector, by dividing by its l^1 norm (in this case the sum of all the coefficients). Finally we can use a standard iterative procedure (the power method or iterative techniques for an equivalent linear system [28, 32, 18, 27]) as an iterative refinement to increase the precision. We remind that computing the PageRank with $c=0.99$ or 1 is very difficult by straightforward techniques, due to slow convergence or even to lack of convergence for $c=1$; see [18] and references therein, and [31, Section 6.1] for a specific discussion on the case $c=0.99$.

All this comprises a new scheme to compute the PageRank, with c equal to 1 or very close to it, is:

- Step 1: Compute $y(c_j)$, $c_j=0.25\cdot\exp(\mathbf{i}2j\pi/p)$, $\mathbf{i}^2=-1$, $j=0,\ldots,p-1$ (*Evaluation via vector DFT*).

- Step 2: *Vector Extrapolation* at the desired (difficult) $c\approx 1$ (e.g. $c=0.85$, $c=0.99$, $c=1$) to obtain $\hat{y}(c)$.

- Step 3: *Project* $\hat{y}(c)$ into the nonnegative cone and do l^1 *normalization.*

- Step 4: Apply *Iterative Refinement* by classical procedures. Since $c \approx 1$, it is advisable to use preconditioning and Krylov techniques, see [18].

We finally remark that the complex Google setting implicit in Section 2.3 is useful not only for matrix theoretic purposes, but also for computation; all the needed formulae (also those in Theorem 2.4.1, see also [39]) are well defined for c in the open unit disk and in a proper disk around $c = 1$. In fact it will be interesting to see whether an algorithm that exploits complex parameters will work well in practice and will enhance the numerical stability as expected. The results of numerical experiments for n of moderate size have been promising. See also [14] for a successful numerical experimentation with real parameters.

A second simpler and maybe more promising possibility comes from looking at the power series in (2.13). The idea is the following: we can zero out the first–order term by forming $y(+\gamma) + y(-\gamma)$. We can zero out the first and second order terms by forming this sum with γ replaced by $\pm i\gamma$, and so on. This looks appealing, but the practical problem is that it requires solution of some large linear systems in a parameter range where the power method diverges. The equations certainly have solutions and can be computed by Krylov techniques (see [18]), but they cannot be obtained by the power method.

2.5.1 Comments on the "ideal" PageRank vector $\widetilde{y}$

First we look at the PageRank problem as an ill–posed problem and we draw some analogies with another famous case of ill–posedness, i.e., the image restoration problem [3, Chapter 1]. Then we provide an interpretation on the vector $\widetilde{y}$, the limit as c tends to 1 of our regularized solutions $y(c)$.

When one considers the pure Google matrix with $c = 1$, i.e., problem (2.3), finding the PageRank (that is, a nonnegative left 1–eigenvector whose entries sum to one) is an ill–posed problem (according to Hadamard [20, Section 2, p. 31]): infinitely many solutions exist and they can all be described as convex combinations of basic nonzero, nonnegative vectors $Z[i]$, $i = 1, \ldots, m$ [39, Section 4], where m is the multiplicity of the eigenvalue 1 of G, i.e., the number of irreducible components of the Markov chain represented by G (see e.g. [23]). These basic vectors are somehow *local* or sparse in the sense that they have a huge number of zero entries: in fact, the reason of such a locality relies on the fact that any $Z[i]$, $i = 1, \ldots, m$, is associated with a single irreducible component of G. On the other hand, when we consider instead $G(c)$ with a parameter $c \in [0,1)$ (or c in the complex open unit disk), we make a sort of regularization that forces stability of the associated numerical problem and uniqueness of the solution. Furthermore, just as in the image restoration problem, our ill–posed problem requires nonnegativity of the solution: in this direction, we may ask if classical procedures used to solve the image restoration problem can be adapted to the PageRank computational problem. Indeed, concerning the algorithm sketched in Section 2.5, we already exploited this similarity in the regularization Step 1 and in the limit process in Step 2, while we borrowed Step 3 again from standard image restoration techniques. Pushing further this reasoning, we may ask in addition if the SPAM pages [31, Section 9.2] can be considered as a noise disturbance, whose effect has to be diminished or eliminated.

Finally let us briefly mention some features of the vector $\widetilde{y}$. Indeed, in the limit as c tends to 1, we obtain a special convex combination of nonnegative solutions, but it is much *less* local: it has a larger support (i.e. the set of indices related to nonzero entries), which clearly depends on the personalization vector v, since $\widetilde{y} = Nv$ with N being the Cesaro averaging projector. For the modeler, this is a good thing, since all of the Web is taken into account, not just a smaller irreducible subset as in the local vectors $Z[i]$, $i = 1, \ldots, m$. The nature of the dependence of the support on v is not yet completely understood and deserves further investigation. However, even the vector $\tilde{y}$ in the real Web shows still a huge number of components with zero ranking; not only this, but many of these pages with zero PageRank are quite important according to common sense, see [9] and the discussion and the new proposals of Section 2.2.

2.5.2 A plain alternative for computing $\widetilde{y} = \lim_{c \to 1} y(c)$

Here we make a plain algebraic modification of the matrix G in such a way that the set of solutions identified by (2.3) remains the same, but the power method converges unconditionally.

The main idea is to modify the row stochastic Google matrix G via a convex sum with the matrix I and more precisely for $\delta \in (0,1)$ we set

$$G_\delta = \delta G + (1-\delta)I.$$

We apply the power method to this new matrix G_δ, that has $\lambda_1(\delta) = 1$ as spectral radius and eigenvalue of (geometric and algebraic) multiplicity m and $|\lambda_j(\delta)| < 1$ for every $m+1 \le j \le n$. Therefore we will observe convergence with an asymptotical rate given by

$$\max_{j \in \{m+1,\ldots,n\}} |\lambda_j(\delta)| < 1. \tag{2.29}$$

Of course the strictly dominating eigenvalue 1 will have an algebraic multiplicity and geometrical multiplicity $m \ge 1$ as in (2.3). So the power method will give back an eigenvector that is function of the initial choice x_0, but, may be surprisingly, not depending on the parameter δ.

An important question arises: how to choose x_0 for which the solution of the power method applied to G_δ coincides with $\widetilde{y} = \lim_{c \to 1} y(c)$?

The interesting fact is that $G_\delta = N^T + R_\delta$ where any eigenvalue of R_δ is of the form $1 - \delta + \delta\lambda_j$, $j = m+1, \ldots, n$, N is the nonnegative projector given in (2.21) and previously described, and the λ_j's are the eigenvalues of the pure Google matrix G. We know that $|\lambda_j| \le 1$ and $\lambda_j \ne 1$ for $j = m+1, \ldots, n$. Hence for any $\delta \in (0,1)$ we have $|1 - \delta + \delta\lambda_j| < 1$ for $j = m+1, \ldots, n$. Consequently the unique solution of the power method applied to G_δ^T with starting vector x_0 is exactly Nx_0. We notice that if x_0 is strictly positive then every iterate is also strictly positive but many of the entries of the limit vector could be zero. Therefore for computing numerically $\widetilde{y} = \lim_{c \to 1} y(c) = Nv$ it is sufficient to set $\delta \in (0,1)$ and to apply the power method to G_δ^T with initial guess v. As already observed the convergence is unconditional, but the speed of convergence depends on δ. In conclusion a proper choice of δ for maximizing the convergence rate of the power method is an interesting issue that we discuss in the next Subsection.

2.5.3 Rate of convergence of the power method

As we will see the matrix G_δ with eigenvalues $\lambda_j(\delta)$ is such that the power method shows a rate of convergence given by (2.29): now we allow the value $\delta = 1$, i.e., we consider $\delta \in (0,1]$. The question is which eigenvalue $\lambda_j(\delta)$ is of maximal modulus for $j \in \{m+1,\ldots,n\}$, with n size of G_δ, and how to choose δ in order to maximize the rate of convergence. We know that for $\delta = 1$ every eigenvalue $\lambda_j(1) = \lambda_j$ that lies on the unit circle in the complex plane has a maximal modulus. In this case there is no convergence since $|\lambda_j| = |e^{i\varphi}| = 1$ for some $j \geq m+1$, $\varphi \in \mathbb{R}$.

In general for $\lambda \in \{\lambda_{m+1},\ldots,\lambda_n\}$ we set $\lambda = re^{\mathbf{i}\varphi}$ and then, since $\lambda_j \neq 1$ for $j \in \{m+1,\ldots,n\}$ (i.e. we cannot have simultaneously $r = 1$ and $\cos(\varphi) = 1$), we find

$$\lambda_j(\delta) = \delta\lambda_j + 1 - \delta = \delta r\cos(\varphi) + 1 - \delta + \mathbf{i}\delta r\sin(\varphi).$$

Hence

$$\begin{aligned}
|\lambda_j(\delta)|^2 &= \delta^2r^2\cos^2(\varphi) + \delta^2r^2\sin^2(\varphi) + (1-\delta)^2 + 2\delta r(1-\delta)\cos(\varphi) \\
&= \delta^2r^2 + (1-\delta)^2 + 2\delta r(1-\delta)\cos(\varphi) \\
&\leq_{\text{setting } \cos(\varphi)=1} \delta^2r^2 + (1-\delta)^2 + 2\delta r(1-\delta) \\
&= (\delta r + 1 - \delta)^2 \\
&\leq_{\text{setting } r=1} (\delta + 1 - \delta)^2 = 1
\end{aligned}$$

and where equality to 1 is impossible since we cannot have at the same time $r = 1$ and $\cos(\varphi) = 1$. This proves the unconditioned convergence of the power method for $\delta \in (0,1)$.

This result can be seen also graphically (ref Fig. 2.2) since for $\delta \in (0,1)$ all the eigenvalues $\lambda_j(\delta)$ with $j \in \{m+1,\ldots,n\}$ lie in the disc with boundary given by $\mathscr{C}_\delta/\{1\}$.

Now the question is how to maximize the rate of convergence i.e. how to choose $\delta \in (0,1]$ for minimizing $s(\delta) = \max_{j\in\{m+1,\ldots,n\}} |\lambda_j(\delta)|$. This translates into a typical min–max problem:

$$\hat{g} \equiv \min_{\delta\in(0,1]} \max_{j\in\{m+1,\ldots,n\}} \left[\delta^2r_j^2 + (1-\delta)^2 + 2\delta r_j(1-\delta)\cos(\varphi_j)\right] \tag{2.30}$$

with $\lambda_j(1) = \lambda_j = r_j e^{\mathbf{i}\varphi_j}$. Indeed for $\delta = 0$, $G_\delta = I$ and therefore $s(0) = 1$ so that the minimum exists in the set $(0,1]$ and is located in the open set $(0,1)$ if, as it usually happens for large Web matrices, at least one r_j equals 1 for $j \geq m+1$.

Looking at the function $\delta^2 r_j^2 + (1-\delta)^2 + 2\delta r_j(1-\delta)\cos(\varphi_j)$ as a function of the radius r_j we notice that it is increasing for $\delta \in (0,1)$ and $\cos(\varphi_j)$; moreover, setting $x_j = r_j \cos(\varphi_j)$ the real part of λ_j, the same function $\delta^2 r_j^2 + (1-\delta)^2 + 2\delta(1-\delta)x_j$ as a function of x_j is increasing again for $\delta \in (0,1)$. Therefore, if $\bar{x}$ is the maximal real part of the eigenvalues λ_j, $j = m+1, \ldots, n$, then it is evident that $|\bar{x}| < 1$ and

$$f_{\bar{x}}(\delta) \equiv \delta^2 + (1-\delta)^2 + 2\delta(1-\delta)\bar{x} \geq \max_{j \in \{m+1,\ldots,n\}} \delta^2 r_j^2 + (1-\delta)^2 + 2\delta r_j(1-\delta)\cos(\varphi_j)$$

so that, by minimizing $f_{\bar{x}}(\delta)$ with respect to δ, we find $\delta_{\text{opt}} = 1/2$ and the upper–bound

$$\hat{g} \leq f_{\bar{x}}(1/2) = \frac{1}{2}(1+\bar{x}) \in [0,1). \tag{2.31}$$

Finally, if the eigenvalue of G_δ coming from that of G with maximal real part is the one maximizing $\delta^2 r_j^2 + (1-\delta)^2 + 2\delta(1-\delta)x_j$ over $j = m+1, \ldots, n$, then we can give interesting lower bounds that is

$$\hat{g} \geq \frac{1}{4}(1 + 2\bar{x} + \bar{r}^2) \geq \frac{1}{4}(1+\bar{x})^2 \tag{2.32}$$

and

$$\hat{g} \geq \frac{\bar{r}^2 - \bar{x}^2}{1 + \bar{r}^2 - 2\bar{x}}, \tag{2.33}$$

where $\bar{r} \in [\bar{x}, 1]$ is the modulus of the eigenvalue real part equal to $\bar{x}$ and $\bar{\varphi}$ its angle.

This last relation it is obtained evaluating

$$\frac{\partial}{\partial \delta}\left(\delta^2 \bar{r}^2 + (1-\delta)^2 + 2\delta\bar{x}(1-\delta)\right) = 0 \tag{2.34}$$

and substituting the result

$$\delta = \frac{1-\bar{x}}{1+\bar{r}^2 - 2\bar{x}} \quad \text{for} \quad \cos(\overline{\varphi})^2 \neq 1 \tag{2.35}$$

in the square modulus of the eigenvalue with maximal real part, that becomes exactly $(\bar{r}^2 - \bar{x}^2)/(1+\bar{r}^2 - 2\bar{x})$.

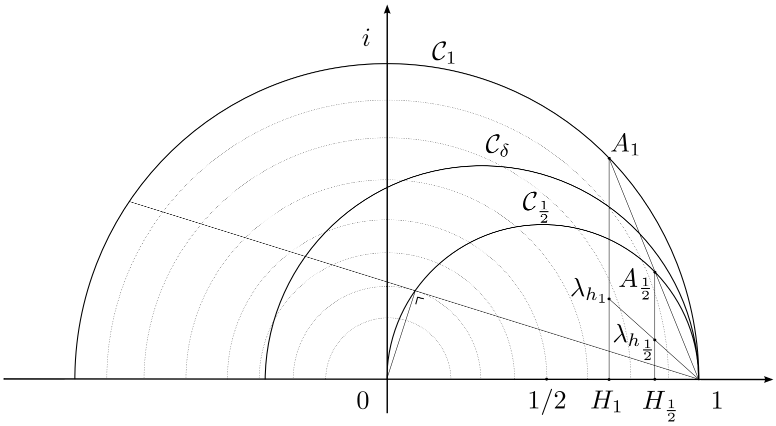


Figure 2.2: Geršgorin circles of G_δ for $\delta \in [0,1]$

These results can also be represented graphically. We start plotting in the complex plane $\mathscr{C}_1$ i.e. the upper half boundary of the Geršgorin region of matrix G (ref Fig. 2.2), since G is real we have specularity with respect to the real axis (more details about Geršgorin regions can be found, for example, in [24, Section 6.1]). If we consider the matrix G_δ for $\delta \in (0,1)$ we get the circle $\mathscr{C}_\delta$. Finally, when $\delta = 0$ the circle collapses in the point 1 on the real axis.

We can observe that for $\delta \in (0,1)$ the trajectory of the generic eigenvalue of G_δ, $\lambda_j(\delta)$ with $j \in \{m+1,\dots,n\}$, is given by the convex combination of the two vectors $[r_j\cos(\varphi_j), r_j\sin(\varphi_j)]^T$ and $[1,0]^T$. It is plain that the minimal modulus of any of these eigenvalues is achieved for the unique δ such that the single trajectory intersects the circumference of radius 1/2 centered in (1/2,0). This coincides with $\mathscr{C}_{1/2}$, i.e. boundary of the Geršgorin region associated with G_δ for $\delta = 1/2$, that is given by the points satisfying the relation $r = \cos(\varphi)$.

As δ varies in the interval $[0,1]$, it is straightforward that the relative position of the eigenvalues does not change. This means that if we suppose to know, for a particular value of $\delta \in (0,1]$, the eigenvalue with maximal real part, $\lambda_h(\delta)$ with $h \in \{m+1,\dots,n\}$, this will remain always the one with maximal real part for every $\delta \in [0,1]$.

Interesting enough we observe that given $\delta \in [0,1/2]$ the problem stated in (2.30) (i.e. minimize, with respect to δ, the maximal modulus of the eigenvalues $\lambda_j(\delta)$ of G_δ for $j \in \{m+1,\dots,n\}$) becomes simply $\hat{g} \equiv \max_{j\in\{m+1,\dots,n\}} 1/4[r_j^2+1+2r_j\cos(\varphi_j)]$ since for δ decreasing from $1/2$ to 0 every eigenvalue increases its modulus. Hence we can restate (2.30) as

$$\hat{g} \equiv \min_{\delta\in\left[\frac{1}{2},1\right]} \max_{j\in\{m+1,\dots,n\}} \left[\delta^2 r_j^2+(1-\delta)^2+2\delta r_j(1-\delta)\cos(\varphi_j)\right] \tag{2.36}$$

Now, assuming that we know the eigenvalue $\lambda_h(1)$, except 1 of course, with maximal real part $\bar{x} = \bar{r}\cos(\bar{\varphi})$. We draw in the complex plane the vertical line that passes through this eigenvalue. This line intersects the $\mathscr{C}_1$ in A_1 and the real axis in H_1. For a generic δ we do the same and we get the points A_δ and A_δ and H_δ with real part equal to $\delta\bar{x}+1-\delta$. In particular $A_{1/2}$ has a minimal distance from the origin among all the possible A_δ. The previous considerations allow us to state that the quantity $\hat{g}$

$$\left|\overline{OH}_{1/2}\right|^2 \leq \hat{g} \leq \left|\overline{OA}_{1/2}\right|^2 \tag{2.37}$$

i.e. relations (2.31) and (2.32).

If we suppose that the trajectory, in function of δ, of the eigenvalue with maximal real part λ_h intersects the circumference $\mathscr{C}_{1/2}$ in the point B, we can rewrite relation (2.33) as

$$\hat{g} \geq \left|\overline{OB}\right|^2 \tag{2.38}$$

Hence if λ_h lies inside the circle $\mathscr{C}_{1/2}$ the above relation becomes simply $\hat{g} \geq \bar{r}^2$. In fact if we impose in (2.35) that $\delta \in [0,1]$ we get $\bar{r} \geq \cos(\bar{\varphi})$ i.e. the relation (2.33) valid when λ_h lies outside circle $\mathscr{C}_{1/2}$.

We observe that in the light of the model proposed in Section 2.2, all these reasonings hold. We add only that in this last case we do not have roots of unity among the λ_j for $j \in \{m+1,\dots,n\}$ since the graph associated with the matrix G can be reduced into the direct sum of irreducible and primitive blocks (ref Section 3.1). This implies that relation (2.31) becomes a strict inequality and that the power method will converge even in the case of $\delta = 1$, but of course not necessarily with a maximal rate of convergence.

Furthermore, we observe that $G_\delta = \delta G + (1-\delta)I$, $\delta \in (0,1]$, is a linear polynomial of G with the condition that the eigenvalue 1 is a fixed point of the transformation and the coefficients are nonnegative. If instead of G_δ we consider any polynomial $p(G)$ of G with nonnegative coefficients and such that $p(1) = 1$, then we could have a larger degree of freedom for maximizing the convergence rate but, of course, the already difficult min–max problem (2.30) would become analytically very intricate. This and other issues such as a more careful study of the min–max problem (2.30) will be the subject of future researches.

2.6 Some comments about prior work

The eigenvalues of the standard real parametric Google matrix $G(c)$ were analyzed by Haveliwala and Kamvar [23] (only the second eigenvalue), Eldén [19], and Langville and Meyer [31] (their proof is the same as that of Reams). A different approach via the characteristic polynomial is suggested in [35, Problem 7.1.17, p. 502]. These authors were apparently unaware of the prior work of Brauer [11] and Reams [38].

Relying on sophisticated results about Markov chains, [39] gives an analysis of the Jordan canonical form of the standard real $G(c)$; it also gives a rational representation for $y(c)$ and computes its limit as $c \to 1$, again in the standard real case only. The Maclaurin series for $y(c)$ was studied in [9], where the partial sums of (2.12) for nonnegative real v and $0 < c < 1$ were identified as the iterates obtained in solving $y^T G(c) = y^T$ with the power method starting at v. Finally, comparing our findings with the results in [9, 30], one important message of the present Chapter is that the point $c = 1$ is not a singularity point for $y(c)$, and hence limits and conditioning of $y(c)$ can be derived and safely handled, both in theory and in practical computations.

2.7 Concluding remarks and future work

In this Chapter, more specifically in Sections 2.1–2.2, we described the original Google model and we discussed its adherence to the reality. Pathologies and limitations of the actual model were pointed out and we proposed some possible improvements which allow both to remove the old pathologies, without introducing new ones, and to preserve the efficiency of the original model. As a matter of fact this new model, besides being useful in Web ranking and search engine optimization, could be of interest also in political/social sciences in ranking, for instance, who/what is influential and who/what is not, in ranking the importance of a paper and/or of a researcher looking in scientific databases (see [4]) etc.

Moreover, in Sections 2.3–2.5, we presented the following results:

1. the eigenvalues of $G(c)$, $\forall c \in \mathbb{C}$;

2. the canonical forms of $G(c)$, $\forall c \in \mathbb{C}$ such that $|c| < 1$ (as a matter of fact the condition $|c| < 1$ can be replaced by the less restrictive (*): $\forall c \in \mathbb{C}$ such that $c\lambda_j \neq 1,\ j = 2,\dots,n$, being $\lambda_1 = 1, \lambda_2, \dots, \lambda_n$ the eigenvalues of $G = G(1)$);

3. a rational expansion for $y(c)$, $\forall c \in \mathbb{C} : |c| < 1$ (in fact only (*) is required);

4. for $c = 1$ the problem (2.3) is ill–posed, but

$$\lim_{\substack{c \to 1 \\ (*)\text{ holds}}} y(c) = \lim_{\substack{c \to 1^- \\ c \in \mathbb{R}}} y(c) = \widetilde{y}$$

   and $\widetilde{y}$ is a solution of (2.3);

5. for this special solution $\widetilde{y}$ we showed that it coincides with Nv where v is the personalization vector and N is a nonnegative projector coinciding with the Cesaro mean $\lim_{r\to\infty} \frac{1}{r+1} \sum_{j=0}^{r} G^j(1)$: the result is due to Lasserre, who calls N the ergodic projector, in a context of probability theory [33], and is known in the field of Web searching engines thanks to [8];

6. if we set $y(1) = \widetilde{y}$, then $y(c)$ is analytic in a proper neighborhood of 1 and its sensitivity $\kappa(y(c),\delta) = \frac{\|y(\tilde{c})-y(c)\|}{\|(y(c)\|}$, $\tilde{c} = c(1+\delta)$, δ complex parameter of small modulus, is defined by the quantity κ_c where $\kappa(y(c),\delta) = \kappa_c|c\delta|(1+O(\delta))/\|y(c)\|$ with respect to a generic induced

norm $\|\cdot\|$; moreover, in a proper neighborhood of $c = \mu^{-1}$, μ belonging to the spectrum of $G = G(1)$ and up to a function independent of c, the factor κ_c grows generically as

$$\max_{\mu\neq 1, \mu\in\sigma(G(1)), n(\mu)\in S(\mu)} F(\mu,c), \tag{2.39}$$

$$F(\mu,c) = \left| z_1(1-c\mu)^{-n(\mu)} + z_2(1-c\mu)^{-n(\mu)-1}(1-c) \right| |c|^{n(\mu)-2}, \tag{2.40}$$

with $z_1 = (n(\mu)-1)(1-c)-c$, $z_2 = c\mu n(\mu)$, $\sigma(W)$ denoting the spectrum of a given square matrix W and $S(\mu)$ denoting the set of all possible sizes of the Jordan blocks related to the eigenvalue μ.

7. numerical procedures of extrapolation type, based on the third item, for the computation of $y(c)$, when c is close or equal to 1 (i.e. the limit Cesaro vector $\widetilde{y}$ of the fourth item and fifth item).

The results in 1) follows from Brauer's Theorem. We discussed the proof and we provided a short historical account in Section 2.6. Findings 2)–7) were obtained in the more general setting of special rank one perturbations studied in the previous Chapter. More specifically, instead of $G(c)$ we considered $A(c) := cA + (1-c)\lambda x v^*$ where $Ax = \lambda x$, $v \in \mathbb{C}^n$, $c \in \mathbb{C}$, and $v^* x = 1$. It is clear that our setting is a special instance of the latter: for example, the existence of the limit as c tends to 1 via any complex path requires that the eigenvalue λ is semisimple and this is the case in the Google setting. However it is important to stress that this generality allows to clarify and even to simplify the mathematical reasoning and the proofs of the results.

Here we report some further comments.

- The results in items 3)–4) were obtained in [39] with the constraint that $c \in [0,1)$. Moreover, item 4) shows that the parameter c acts like a regularization parameter that stabilizes problem (for a general treatment of regularization techniques see [20]): the nice thing is that, as c tends to 1, we obtain a limit vector $\widetilde{y}$, one of the solutions of the original problem.

- The above analysis can be attacked also by transforming the PageRank problem into an equivalent linear system formulation (see e.g. [31]).

In that case we write $y(c) = (1-c)(I - cG^T)^{-1}v$ and indeed this is a compact version of formula (2.21), used in Section 2.4 for a detailed analysis of $y(c)$ and of its conditioning as a function of the parameter c.

- The algorithms that we proposed are new and are partly based on specialized extrapolation procedures discussed in [15, 13]; moreover, we should benefit from the choice of a complex parameter thanks to items 2)–4) with respect to [14], especially in terms of stability.

- Another important issue is how to interpret the computed vector $y(c)$ when c is equal or close to 1 and how to distinguish it from the infinitely many solutions existing when $c = 1$.

As a final remark, we stress that the analysis of our matrix–theoretic oriented approach is also valid for the modified enhanced models proposed e.g. in [1, 41] etc. or discussed here in Section 2.2. Indeed the interest in the general matrix–theoretic analysis relies on its level of adaptability. In fact the results and the conclusions of Sections 2.3–2.4 are virtually unchanged if one considers a different way of handling nodes or if one allows self–links giving raise to a different definition of $G(1)$. Moreover, in this context we must no forget that there exist completely different applications [6, 29] including dynamical agents theory [34, 43], where the idea and the computational suggestions in Section 2.5 have a lot of potential to be further developed and studied.

Part II

Stochastic families The consensus problem

Chapter 3

Coordination of autonomous agents

The modelling and analysis of groups of autonomous agents and their coordination have attracted a growing number of researchers from different fields like physics, biology, engineering and mathematics for a couple of decades. This is partly due to the existence of several applications of this subject in different areas like cooperative control and formation of unmanned air vehicles (UAVs), flocking and schooling of biological and artificial agents, attitude alignment of clusters of satellites, collaborative mobile robots, sensor networks and congestion control in communication networks [103, 50, 89, 46, 47].

Many models have been proposed (see [103, 46] and references therein), nevertheless in this Chapter we focus our attention on a particular one which is a discrete–time linear protocol, proposed by Vicsek et al. in 1995 [50], which allows to model autonomous agents and their tendency to agree each other through a distributed decision making process.

This tendency is typical of a wide range of biological swarming systems like herds of quadrupeds, schools of fish, flocks of flying birds, bacterial colonies [50] and can be found, in addition, in many artificial systems like groups of unmanned aerial vehicles, mobile robots or networks of sensors [103, 46].

For this model we analyze the convergence of the solution to a global consensus and, following Jadbabaie et al. [89], we give sufficient conditions, based on the spectral properties of the family of stochastic matrices associated with the system, which guarantee this convergence. The model proves to be related to the PageRank one and its analysis is based on algorithms, developed for generic families of matrices, which we present in the next part of the thesis.

This Chapter develops, more precisely, in the following way:

Section 3.1 is devoted to present notation, definitions and properties about directed graphs. In Section 3.2 we recall the Vicsek model, its properties, a few theoretical results and we describe how, making use of the so–called joint spectral radius defined in the next Chapter, we can guarantee for the system to reach a global consensus. In the last part of the Section we discuss the connections between the Vicsek and the Google model and we present a few simulations. In Section 3.3 we study the convergence to a global consensus of componentwise weakly connected networks with fixed topology and we give alternative proofs for a few Theorems presented in the literature. The Chapter concludes with final remarks and future work.

3.1 Notation and definitions

We start with a quick survey of basic definitions and notions used in this Chapter.

Given two sets A and B, with the expression $B\backslash A$ we denote the set–theoretic difference of B and A, i.e. the set of elements which are in B, but not in A

$$B\backslash A = \left\{ x \in B \,\middle|\, x \notin A \right\}. \tag{3.1}$$

3.1.1 Definition (Permutation matrix). *A matrix $P \in \mathbb{R}^{n\times n}$, is called a permutation matrix if exactly one entry in each row and column is equal to* 1*, and all other entries are* 0*.*

For every permutation matrix $P^T P = P P^T = I$ i.e. $P^{-1} = P^T$ [24].

Note that from now on we make use of the letter e to represent the column vector of all ones, $e = [1\ 1\ \cdots\ 1]^T$.

3.1.2 Definition (Digraph)**.** *A* weighted directed graph $\mathscr{G}$ *(shortly* weighted digraph*) of order n is uniquely defined by $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$, where*

$\mathscr{V}$ *is the set of n vertices/nodes of the graph $\mathscr{V} = \{v_1, \ldots, v_n\}$.*

$\mathscr{E}$ *is the set of directed edges between nodes of the graph such that $\mathscr{E} = \{e_{ij}\} \subseteq \mathscr{V} \times \mathscr{V}$, with e_{ij} the edge given by $e_{ij} = (v_i, v_j)$.*

A *is the* weighted adjacency matrix $A = [a_{ij}]_{i,j=1}^{n}$, *where each entry is called* weight *or* cost. *The structure of A derives from $\mathscr{E}$ in the following way: the entry a_{ij} is a positive value (cost or weight) if the corresponding edge e_{ij} belongs to the set $\mathscr{E}$, 0 otherwise.*

It is possible to define also a digraph without specifying the weights. In this case we have simply a so–called digraph *which is identify using the set of vertices $\mathscr{V}$ and edges $\mathscr{E}$, $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. In this latter case the weights are all $1's$ and $0's$, therefore, the adjacency matrix $A = [a_{ij}]$ is such that $a_{ij} = 1$ if the directed edge e_{ij} between v_i and v_j belongs to $\mathscr{E}$, 0 otherwise.*

The digraph is simple *if there are no self–loops ($a_{ii} = 0$ for all $i = 1, \ldots, n$) and repeated edges ($\mathscr{E}$ contains only distinct elements i.e. there is no more than one directed edge between any two different and ordered vertices).*

The neighborhood *of a node/vertex v_i is denoted by N_i and it is defined as the set of the indices of all the nodes with which v_i is linked to*

$$N_i = \left\{ j \in \{1, \ldots, n\} \;\middle|\; (v_i, v_j) \in \mathscr{E} \right\} \tag{3.2}$$

the corresponding nodes are called neighbors *of v_i.*

A path *in $\mathscr{G}$ is a sequence of vertices such that from each of these vertices there is an edge to the next vertex in the sequence.*

We mention for completeness that an *undirected graph* or simply *graph* is defined exactly like a digraph, but with edges that are all undirected. So the adjacency matrix A associated with an undirected graph is always symmetric and the ordering of the nodes in an edge does not matter. In an undirected graph $\mathscr{G}$, two vertices v_i and v_j are *connected* if $\mathscr{G}$ contains a path from v_i to v_j, otherwise they are *disconnected*. If the two vertices are additionally connected by a path of length 1, i.e. by a single edge, the vertices are defined *adjacent*. An undirected graph is said to be *connected* if all the vertices in the graph are connected each other. A *connected component* of an undirected graph $\mathscr{G}$ is a maximal connected subgraph of $\mathscr{G}$. Each vertex and each edge belongs to exactly one connected component of $\mathscr{G}$.

Instead, a digraph $\mathscr{G}$ is *strongly connected* if and only if for every ordered pair of nodes (v_i, v_j), with $i \neq j$, there exists a sequence of directed edges, a so–called directed path, leading from v_i to v_j.It is *weakly connected* if replacing all of its directed edges with undirected edges produces a connected (undirected) graph. A *strongly connected component* or *strong component* is a maximal strongly connected subdigraph of $\mathscr{G}$ i.e. a maximal subset of strongly connected nodes. A digraph is *componentwise weakly connected* if all the strong components are only weakly connected each other. From now on we consider always digraph, if not differently stated.

Given $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ the subset $\mathscr{X} \subseteq \mathscr{V}$ is a *stable set* if $\forall v_i \in \mathscr{X}$ it does not exist any edge $e_{ij} \in \mathscr{E}$ such that $v_j \in \mathscr{V} \backslash \mathscr{X}$. A stable set $\mathscr{X}$ is *minimal* if it does not exist any stable set $\mathscr{Y}$ such that $\mathscr{Y} \subset \mathscr{X}$ or, equivalently, it is minimal if by passing any node from $\mathscr{X}$ to $\mathscr{V} \backslash \mathscr{X}$ the remaining set $\mathscr{X}$ is no more stable.

Given $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ and $\mathscr{Y} \subseteq \mathscr{V}$, the *subdigraph* induced by $\mathscr{Y}$ on $\mathscr{G}$ is $\widetilde{\mathscr{G}} = (\mathscr{Y}, \mathscr{E}_{\mathscr{Y}}, A_{\mathscr{Y}})$ where $\mathscr{E}_{\mathscr{Y}} = \left\{ e_{ij} \in \mathscr{E} \;\middle|\; v_i, v_j \in \mathscr{Y} \right\}$, and $A_{\mathscr{Y}}$ is the submatrix formed by selecting the i rows and columns of A with i indices of the nodes in the subset $\mathscr{Y}$.

It is easy to prove the following Lemmas

3.1.3 Lemma. *Each stable set contains minimal stable sets. The intersection of stable sets is stable. Two minimal stable sets are always disjoint.*

3.1.4 Lemma ([44, Lemma 2])**.** *Given $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ with $\mathscr{X} \subseteq \mathscr{V}$ a minimal stable set, then the subdigraph induced by $\mathscr{X}$ on $\mathscr{G}$ is strongly connected.*

3.1.5 Definition. *Given $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ and a node $v_i \in \mathscr{V}$, we define its in–degree* $\deg_{\text{in}}(v_i)$ *and out–degree* $\deg_{\text{out}}(v_i)$ *as*

$$\deg_{\text{in}}(v_i) = \sum_{j=1}^{n} a_{ji} \qquad \text{and} \qquad \deg_{\text{out}}(v_i) = \sum_{j=1}^{n} a_{ij} \tag{3.3}$$

So the in–degree is related to the set of ingoing edges to node v_i while the out–degree to the outgoing ones.

For $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, digraph with adjacency elements–weights 0/1, we have that $\mathrm{deg}_{\mathrm{out}}(v_i) = |N_i|$, where $|N_i|$ is the cardinality of N_i.

3.1.6 Definition. *The* degree matrix *of the weighted digraph $\mathscr{G}$ is a diagonal matrix*

$$D = \begin{bmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{bmatrix} \tag{3.4}$$

where $d_{ii} = \mathrm{deg}_{\mathrm{out}}(v_i)$ for every $i \in \{1, \ldots, n\}$.

3.1.7 Definition. *The* graph Laplacian *associated with the weighted simple digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ is defined as*

$$L = D - A. \tag{3.5}$$

with D the degree matrix of $\mathscr{G}$.

We observe that, by definition, every row sum of the graph Laplacian matrix is zero, therefore, the Laplacian matrix always has a zero eigenvalue corresponding to the right eigenvector e and $\mathrm{rank}(L) \leq n-1$.

3.1.8 Definition. *The* Perron matrix P *of a weighted simple digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ is defined as*

$$P = I - \Delta t L. \tag{3.6}$$

with L the graph Laplacian of $\mathscr{G}$ and Δt the step–size or time interval.

Let us consider values $\{\theta_i\}_{i=1}^n$, associated with each node/agent i, which represent a physical quantity like position, direction, temperature, voltage etc.

We define the pair $(\mathscr{G}, \theta)$ as an *algebraic graph* or *network* where $\theta \in \mathbb{R}^n$ and the *topology* is given by the weighted digraph $\mathscr{G}$.

We say that two nodes, v_i and v_j, agree in a network if and only if $\theta_i = \theta_j$. The nodes of a network reach a *global consensus* if and only if $\theta_i = \theta_j$ for every $i, j \in \{1, \ldots, n\}$.

As recalled in Section 1.1 a square matrix A is *row–stochastic* if it has real nonnegative entries and each row entries add up to 1; A is *column–stochastic* if A^T is row–stochastic. We say that A is *stochastic* if it is either row–stochastic or column–stochastic. The row–stochasticity of A can be formulated also as $Ae = e$, hence e is a right eigenvector of A associated with the eigenvalue $\lambda = 1$ and $\mathrm{span}\{e\}$ is an invariant subspace of A. It follows from Lemma 2.3.1 that the spectral radius is $\rho(A) = 1$ (ref Section 4.1). It is easy to prove that the sets of row–stochastic and column–stochastic matrices are both closed under multiplication.

3.1.9 Definition (Ergodicity)**.** *Any stochastic matrix A for which $\lim_{i\to\infty} A^i$ is a matrix of rank one is defined to be an* ergodic matrix*.*

3.1.10 Definition (Irreducibility and primitivity)**.** *A nonnegative matrix A is defined* irreducible *if and only if, considering A as the adjacency matrix of a weighted digraph $\mathscr{G}$, $\mathscr{G}$ is strongly connected.*

A nonnegative, irreducible matrix A having only one eigenvalue on its spectral circle is said to be primitive*, only one eigenvalue has the same modulus of $\rho(A)$. If, instead, there are $h > 1$ eigenvalues with modulus equal to $\rho(A)$, A is said to be* imprimitive *or* cyclic *and h is said to be the period of A, or the index of imprimitivity, or also the order of cyclicity .*

It is possible to prove that A is primitive if and only if $A^m > 0$ for some $m > 0$ (Frobenius' Test [35]). We recall also a sufficient condition for the primitivity

3.1.11 Proposition ([35, Example 8.3.3])**.** *If an irreducible nonnegative matrix A has at least one positive diagonal element, i.e.* $\mathrm{trace(A)} > 0$*, then A is primitive.*

The Perron–Frobenius Theorem allows us to state that an irreducible nonnegative matrix A has a unique eigenvalue λ exactly equal to the spectral radius of the matrix, $\lambda = \rho(A)$, and it is the only eigenvalue possessing a right eigenvector with all positive entries, this is said to be the *Perron vector* (see [35, Section 8.3] for a survey on both irreducible matrices and this Theorem). Therefore an imprimitive matrix A, since by definition it is irreducible and nonnegative, has one eigenvalue exactly equal to the spectral radius and $h-1$ eigenvalues with modulus equal to $\rho(A)$.

We conclude the Section proving that

3.1.12 Proposition. *If a matrix A is primitive stochastic, then it is ergodic.*

Proof. It suffices to consider the similarity transformation that reduce A to its Jordan canonical form. Supposed A row–stochastic, otherwise we consider the transpose, we have

$$A = SJS^{-1} = \begin{bmatrix} e & S_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} y^T \\ Z_1^T \end{bmatrix} \tag{3.7}$$

where e and y are the Perron vectors of A and A^T respectively, while B is given by the direct sum of the Jordan blocks associated with all the eigenvalues of A except 1. Since the matrix is primitive stochastic we have that all the eigenvalues of B are strictly less than 1 in modulus. So

$$\lim_{n\to\infty} A^n = S\left(\lim_{n\to\infty} J^n\right) S^{-1} = \begin{bmatrix} e & S_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y^T \\ Z_1^T \end{bmatrix} = e\,y^T \tag{3.8}$$

□

3.2 Vicsek model and the consensus problem

3.2.1 The model

The discrete–time model proposed by Vicsek et al. [50] allows to represent a leaderless coordination system: given n agents/particles, which we identify using a number from 1 to n, that move freely on the plane, we consider at time $t = 0$ that all the agents are randomly distributed, they have the same absolute velocity v and they move in a random direction. The set $N_i(t)$ contains the indices associated with the neighbors of agent i at time t which are all the agents that are inside a circle of radius r, value arbitrarily chosen and equal for all the agents, centered in $[x_i(t), y_i(t)]$, position of agent i at time t. We define $\theta_i(t)$ as the direction/angle in which agent i is moving at time t, with respect to a direction of reference, and we assign the initial direction $\theta_i(0)$, for every $i \in \{1,\dots,n\}$. The evolution of the whole system is considered in discrete time, the time interval Δt, between two consecutive time steps, is fixed for simplicity to 1. $\theta_i(t+1)$ is the updated direction of agent i at time step $t+1$ and is obtained taking a weighted average of its own direction and the directions of its neighbors at time step t, as described in the following *evolution equation*

$$\theta_i(t+1) = \frac{\theta_i(t) + \sum_{j\in N_i(t)} w_{ij}(t)\theta_j(t)}{1+\sum_{j\in N_i(t)} w_{ij}(t)} \quad \text{for} \quad i = 1,\dots,n. \tag{3.9}$$

the denominator is intended for normalization. Values w_{ij} represent the weights we use in averaging, the simplest choice is to consider an not–weighted digraph i.e. all the weights equal 1 or 0.

The position of every agent changes according to the intuitive law

$$\begin{bmatrix} x_i(t+1) \\ y_i(t+1) \end{bmatrix} = \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} + \begin{bmatrix} v\cos(\theta_i(t)) \\ v\sin(\theta_i(t)) \end{bmatrix} \Delta t \tag{3.10}$$

where $\Delta t = 1$. Collisions are not taken into account in this model, in fact, two agents are allowed to occupy the same position at the same time. Furthermore Vicsek et al. added into (3.9) an extra term $\Delta\theta$, a random number which simulate the presence of an external disturbance. We are not going to consider noise terms in our study, but we observe that it is possible to take account of this extra term simply inserting one or more phantom agents/nodes in the system, updating their directions randomly at every step without applying equation (3.9) and considering these phantom agents as neighbors of the actual agents of the system.

As we said the autonomous agents do not follow a leader, they just consider the behavior of their neighbors and change direction consequently. Despite its simplicity, this model is able to represent a nontrivial behavior: the emergence of the so–called *consensus*. For the model just described to reach the consensus means that there is a kinetic phase transition from no net–transport (each agent moves in a random direction determining a mean direction for the system that is around zero) to finite net–transport (the majority of the agents move in the same direction determining a non zero mean direction).

We observe that this model turned out to be a special version of a model introduced a few years before by Reynolds [48] for the visual simulation in the animation industry of flocking and schooling behaviors, but also can be interpreted as a linear approximation of the Kuramoto equation [45] which models a population of oscillators with sinusoidal coupling terms. Coupled oscillators are, for instance, orbiting planets, pacemaker cells in human heart, flashing fireflies; they have several applications in physics and engineering (we suggest the enjoyable popular scientific book on this topic "Sync: The Emerging Science of Spontaneous Order" by Strogatz [49]).

The system of n agents and their connectivity can be described by means of a weighted simple digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$: the agents can be seen as the vertices $v_i \in \mathscr{V}$, the communications between agents are the edges $e_{ij} = (v_i, v_j) \in \mathscr{E}$ and the neighbors of an agent are the neighbors of the corresponding vertex (recall Definition 3.1.2). We observe that the information flow in a flock is typically not bidirectional that is way we consider digraphs for the consensus problem. Since the topology of the network of agents goes through changes that are discrete–event in nature (like links failure and creation or nearest neighbors coupling) we allow for time–dependent communication patterns i.e. we consider a *dynamic graph* $\mathscr{G}(t) = (\mathscr{V}, \mathscr{E}(t), A(t))$ with time–varying adjacency matrix and set of edges. The adjacency matrix $A(t) = [a_{ij}(t)]_{i,j=1}^n$ is such that $a_{ij}(t) = w_{ij}(t)$, for $i, j = 1, \ldots, n$, with w_{ij} the weights which appear in (3.9).

Let us define the *evolution matrix*

$$F_{i_t} = [f_{hk}(t)]_{h,k=1}^n = (D(t) + I)^{-1} (A(t) + I) \tag{3.11}$$

where i_t is an index belonging to the finite or infinite set $\mathscr{I}$, I is the identity matrix and $D(t)$ is the degree matrix associated with the weighted simple digraph (recall Definition 3.1.6). Therefore

$$f_{hk}(t) \quad = \quad \begin{cases} \frac{1}{1+\sum_{j \in N_h(t)} w_{hj}(t)} & \text{if} \quad k = h \\ \frac{w_{hk}(t)}{1+\sum_{j \in N_h(t)} w_{hj}(t)} & \text{if} \quad k \in N_h(t) \\ 0 & \text{otherwise} \end{cases}$$

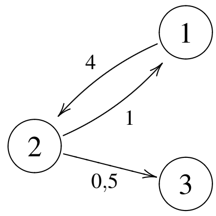
Note that the evolution matrix F_{i_t} is not necessarily symmetric since the graph is directed, but it is row–stochastic by construction.

We can now rewrite the evolution equation (3.9) in matrix form

$$\theta(t+1) = F_{i_t} \theta(t) \tag{3.12}$$

where $\theta(t) = [\theta_1(t), \dots, \theta_n(t)]^T$.

As an example we consider the following toy *simple digraph* with fixed topology which describe, for instance, three sensors that measure the temperature $\theta(t)$ of a chamber and have to agree on a common measure



The adjacency matrix associated with this digraph is

$$A = \begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0,5 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.13}$$

and, consequently, by (3.11) the evolution matrix F is

$$F = \begin{bmatrix} 0,2 & 0,8 & 0 \\ 0,4 & 0,4 & 0,2 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.14}$$

If we choose as initial state $\theta(0)$, the state of the system at time t is given by

$$\theta(t) = F^t \theta(0). \tag{3.15}$$

However, in many practical problems the position of each agent changes following equation (3.10) and, consequently, the relations between the agents change with time . So, in general, we deal with a linear time–varying system: the connectivity $\mathscr{G}(t)$ of the digraph evolves over time and the evolution matrix F_{i_t} is likely to change at every time step. We define $\mathscr{F} = \left\{F_j\right\}_{j \in \mathscr{I}}$ as the set of all possible evolution matrices F_{i_t} associated with the system of autonomous agents.

In order to describe completely this system we should include in our model also equation (3.10) and evaluate F_{i_t} in function of the position in the plane of each agent at time t. While such relation is easy to derive and is essential for simulation purposes, it would be difficult to take into account it in a convergence analysis. That is why, following Jadbabaie et al. [89], we ignore this dependence and assume instead that the system evolves following a random sequence i_t which takes values in the set of indices $\mathscr{I}$. The aim is to find sufficient conditions which ensure a consensus among all the agents

$$\lim_{t\to\infty}\theta(t) = \lim_{t\to\infty} F_{i_t}\cdot F_{i_{t-1}}\cdot\ldots\cdot F_{i_1}\cdot F_{i_0}\theta(0) = e\,\theta_{cons}. \tag{3.16}$$

for any possible sequence of indices $i_t \in \mathscr{I}$ and for any initial set of agent headings $\theta(0)$.

About the fixed topology case, the following Proposition is about the convergence to a global consensus of a directed network with strong connectivity and, in Section 3.3, we present a few Theorems concerning the case of a digraph componentwise weakly connected, always with fixed topology, and we provide for them alternative proofs to those presented in the literature.

3.2.1 Proposition. *Assume $\mathscr{G} = (\mathscr{V},\mathscr{E},A)$ is a not time–varying and strongly connected weighted digraph, consider the associated network of agents with evolution equation given by (3.12) where $F_{i_t} = F$ for every $t \geq 0$. Then, equation (3.12) globally asymptotically solves a consensus problem for any initial set of agent headings $\theta(0)$, i.e. equation (3.16) holds true for $\theta_{cons} = y^T\theta(0)$, where y is the Perron vector of the irreducible and nonnegative F^T.*

Proof. The proof is simple and is based on the fact that matrix F (3.11) is row–stochastic and is the (rescaled) sum of an irreducible and an identity matrix, therefore, by Proposition 3.1.11, it is primitive. Furthermore Proposition 3.1.12 guarantees that

$$\lim_{t\to\infty}\theta(t) = \lim_{n\to\infty} F^n\theta(0) = e\,y^T\theta(0) = e\,\theta_{cons} \tag{3.17}$$

with $\theta_{cons} \in \mathbb{R}$ given by the inner product between y, the Perron vector of F^T, and $\theta(0)$. □

On the other hand, there are also situations in which equation (3.16) does not hold true. For instance when one agent is in an initial position such that it never acquires any neighbors the weighted digraph $\mathscr{G}$ has a vertex which remains always isolated. This condition is highly possible when the density of the system is low or, equivalently, when the radius r, used to define the set of neighbors (ref page 75), is very small. The opposite condition previously cited, a not time–varying and strongly connected weighted digraph, is likely if r is very large (the density is high), in fact, in this case all the agents remain neighbors each other for all the time.

The most complex and therefore interesting situation is that of a dynamic $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$, i.e. a weighted digraph which changes over time. In order to study this last case we first recall once more that all the evolution matrices in $\mathscr{F} = \{F_j\}_{j \in \mathscr{I}}$, which are associated with the system of autonomous agents via equation (3.12), are *row–stochastic* . Furthermore we observe that it is possible to generalize the concept of spectral radius of a matrix to the case of a set/family of matrices. There are different ways to make this generalization, but all coincide with each other in the unique value defined as $\rho(\mathscr{F})$ when the set of matrices $\mathscr{F}$ is finite or bounded. The $\rho(\mathscr{F})$ is called simply *spectral radius* of $\mathscr{F}$ or *joint spectral radius* , which is how Rota and Strang called the first generalization introduced in 1960 [110].

The joint spectral radius gives the maximal growth rate of products of matrices belonging to the set, just as the spectral radius of a matrix provides the maximal growth rate of its powers, and, in addition, allows to give information about the *uniform asymptotic stability* (u.a.s.) of the system, in fact, the system is u.a.s. if and only if $\rho(\mathscr{F}) < 1$ (ref Definition 4.2.1 and Property 8 on page 114).

One of the equivalent generalizations of spectral radius of a matrix, the so–called *generalized spectral radius*, allows to interpret the spectral radius $\rho(\mathscr{F})$ as the $\sup\limits_{k \geq 1} \max_{P \in \mathscr{P}_k(\mathscr{F})} \rho(P)^{1/k}$ i.e. the sup among the maximal values of the normalized spectral radii of all $\mathscr{P}_k(\mathscr{F})$, products of length k generated using matrices in $\mathscr{F}$ (ref Definition 4.2.3 and equation (4.25)). For a survey on the spectral radius of a set of matrices and its properties we refer the reader to Chapter 4.

The joint spectral radius of $\mathscr{F}$ and the row–stochasticity of all the evolution matrices play an essential role in the study of the system. As we recalled in the previous Section the product of stochastic matrices is always a stochastic matrix, hence, the spectral radii of all the possible products of matrices in $\mathscr{F}$ are always equal to 1. This implies, by definition of the generalized spectral radius, that $\rho(\mathscr{F}) = 1$.

Since $\rho(\mathscr{F}) = 1$ the system is never u.a.s., so we know that the products of the evolution matrices never converge to zero or, equivalently, starting from any $\theta(0) \neq 0$ the headings vector $\theta(t)$ never converges to zero. The question we want to answer is, instead, if all the headings of the n agents will eventually be equal to a value θ_{cons} and, thus, moving all together in the same direction.

To answer this question the joint spectral radius proves to be extremely helpful. Jadbabaie et al. in [89] suggest to reduce the set $\mathscr{F} = \{F_j\}_{j\in\mathscr{I}}$ of $n\times n$ evolution matrices to a set $\widetilde{\mathscr{F}} = \left\{\widetilde{F}_j\right\}_{j\in\mathscr{I}}$ of $(n-1)\times(n-1)$–matrices getting rid of the one dimensional common invariant subspace of $\mathscr{F}$ $\mathrm{span}\{e\}$ (recall that e is an invariant subspace for every row–stochastic matrix). Defined the matrix

$$T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & -1 & & 0 \\ \vdots & & \ddots & \\ 1 & 0 & & -1 \end{bmatrix} = T^{-1} \tag{3.18}$$

we consider the similarity transformation

$$TF_jT^{-1} = \begin{bmatrix} 1 & \bigstar \\ 0 & \widetilde{F}_j \end{bmatrix} \quad \forall j \in \mathscr{I} \tag{3.19}$$

and denoted by $\sigma(F_j)$ the *spectrum* of F_j, i.e. the set of all the eigenvalues of F_j, we have that

$$\sigma(F_j) = \sigma(\widetilde{F}_j) \cup \{1\} \quad \forall j \in \mathscr{I} \tag{3.20}$$

where 1 is the eigenvalue associated with the right–eigenvector e of $F_j \in \mathscr{F}$ for every $j \in \mathscr{I}$. Note that the matrices $\widetilde{\mathscr{F}} = \left\{\widetilde{F}_j\right\}_{j\in\mathscr{I}}$ are not stochastic and are not necessarily composed of non–negative elements.

So, the convergence of the generic product $F_{i_t} \cdot F_{i_{t-1}} \cdot \ldots \cdot F_{i_1} \cdot F_{i_0}$ of evolution matrices belonging to $\mathscr{F}$ to a rank–one matrix of the type ey^T, with $y \in$

$\mathbb{R}^n$, is equivalent to the convergence to zero of the corresponding product $\widetilde{F}_{i_t} \cdot \widetilde{F}_{i_{t-1}} \cdot \ldots \cdot \widetilde{F}_{i_1} \cdot \widetilde{F}_{i_0}$. The limit behavior of this last product can be studied evaluating the joint spectral radius of the set $\widetilde{\mathscr{F}}$ which is u.a.s. if and only if $\rho(\widetilde{\mathscr{F}}) < 1$. For this reason the evaluation of the second spectral radius of the set $\mathscr{F}$ allows to have a priori information about the convergence of a system of agents to a global consensus.

In fact the convergence rate of the system, in the worst possible dynamical configuration chosen by the agents, is given by the second spectral radius of the set of matrices $\mathscr{F}$ which is the spectral radius of $\widetilde{\mathscr{F}}$: the more the second spectral radius of $\mathscr{F}$ is similar to 1 the slower, in the worst case, the system reaches a common agreement. In the limit case in which this second spectral radius of the set $\mathscr{F}$ is exactly equal 1 there will products of matrices in $\mathscr{F}$ such that they will be not ergodic (ref Definition 3.1.9) an, thus, the system will not converge to a global consensus. In this last case there will be two or more subsets of agents, corresponding to all the minimal stable sets of the system, that will show a local consensus on the headings, while all the other nodes will have different headings whose values are bounded by the headings of the nodes in the minimal stable sets, like for a fixed topology network which is componentwise weakly connected (ref Section 3.3).

So, checking if $\rho(\widetilde{\mathscr{F}}) < 1$ is sufficient to guarantee that even in the worst case all the headings $\theta(t)_i$ of the n agents will converge to a common value θ_{cons} for $t \to \infty$. The value θ_{cons} depends on the initial state $\theta(0)$ and the effective dynamic of the system.

3.2.2 Connections with the Google model

In Section 2.1 we described the Google model explaining how the matrix G, which is the adjacency matrix associated with the Web considered as a huge directed graph, and equation (2.3) can be interpreted as a translation in formulae of the notion of VIP (very important person). This connection allows to identify the basic PageRank of the Web with the left 1–eigenvector of the row–stochastic matrix G. Now, if we consider a network of autonomous agents with fixed topology $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$, we can associate with the adjacency matrix A both a row–stochastic matrix $F_{i_t} = F$, as it appears in equation (3.11), and a Google matrix G. There are only two tiny differences in the structure of these two matrices: in F, in contrast with G, we include also self–loops and the weights w_{ij}, for all the neighbors $j \in N_i$ of each agent i, can be different each other, as explained in the previous Section. In the particular case of a fixed topology $F_{i_t} = F$ with F irreducible and primitive, if we apply equation (2.3) to F instead of G we obtain the *left* 1–eigenvector of F which corresponds to the PageRank of the network (the so–called Perron vector of F^T). So the solution of the iterative equation (3.12) of the Vicsek model converges, for a strongly connected system, to the unique *right* 1–eigenvector of F, that is the vector e, multiplied by the scalar value θ_{cons} given by the inner product between the PageRank y of F and the initial condition of the system $\theta(0)$ (ref Proposition 3.2.1). Therefore the PageRank analysis proves to be all we need to completely characterize the global consensus of a network of agents with strong connectivity.

When the system of autonomous agents $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ has a fixed topology, but no strong connectivity, the solution of equation (3.12), with $F_{i_t} = F$ for every $t \geq 0$, converges in the limit to $XY^T\theta(0)$, where $\theta(0)$ is the initial state of the system, X and Y are $n \times m$–matrices whose columns are m linearly independent right and left 1–eigenvectors of F, respectively. The product XY^T, continuing the parallelism with the Google model, can be identified with the ergodic projector of F

$$XY^T = N^T = I - (I - F)(I - F)^D,$$

which appears in the context of the PageRank model in equations (1.14) and (2.15). In fact, as we will detail in the next Section, if there is no strong connectivity we can identify m minimal stable subsets of V which correspond, by Lemma 3.1.4, to m strongly connected components of $\mathscr{G}$. Therefore, by Corollary 3.3.2, the eigenvalue 1 of F is semisimple and has multiplicity m.

So, for a fixed topology network, the study of the connectivity of the evolution matrix F and the evaluation of its PageRank prove to be all we need to understand the behavior of the system.

What about the dynamical case? As explained in the previous Section, if we deal with a time–varying network $\mathscr{G}(t) = (\mathscr{V}, \mathscr{E}(t), A(t))$, it is sufficient to check that the second joint spectral radius of the associated set of evolution matrices $\mathscr{F} = \{F_j\}_{j\in\mathscr{I}}$ is strictly less than 1 to ensure the unconditional convergence of $\theta(t)$, solution at time t of equation (3.12), to a global consensus as $t \to \infty$. About the PageRank analysis, in this context the effective PageRank will depend on the specific dynamic of the connections. Nevertheless, the evaluation of the PageRank of each evolution matrix in $\mathscr{F}$ and the estimation of the second joint spectral radius of this set allow us to have interesting information on the system, as detailed in the following.

3.2.3 Experimental results

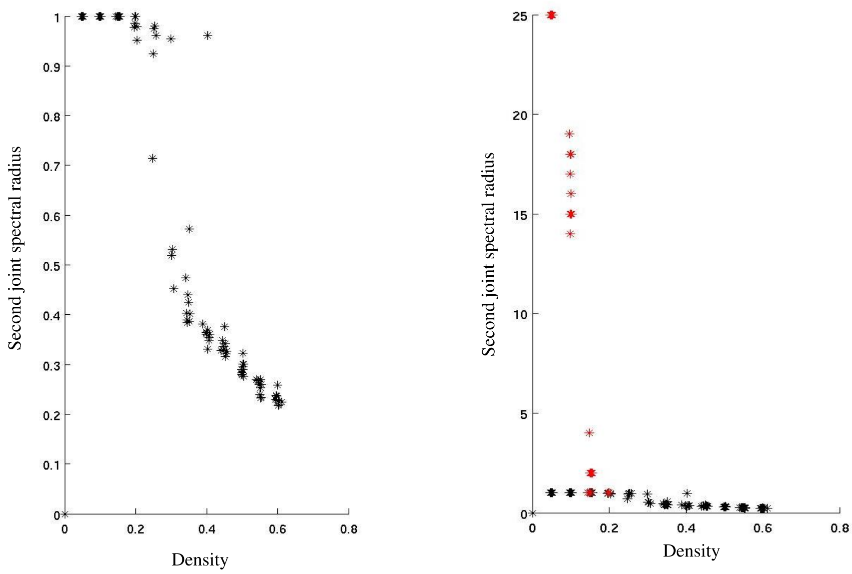
Using Matlab function rand, we have generated sets of random evolution matrices $\mathscr{F} = \{F_j\}_{j=1}^{k}$ and we have studied their second joint spectral radius by means of the algorithm proposed by Gripenberg in [74], which allows to compute lower and upper bounds of $\rho(\mathscr{F})$ and provides candidate spectrum–maximizing products of progressively higher length (ref Chapter 5).

We have studied sets $\mathscr{F}$ with cardinality k and matrix dimension n given by

$$\begin{aligned} k &\in \{5,10,15,20,25,30,35\} \\ n &\in \{5,10,15,20,25,30,75,100,125\}. \end{aligned}$$

In particular, considering the number of neighbors of an agent in terms of density (fixing a value of density ρ we have generated k evolution matrices corresponding to networks in which every agent i has a neighborhood N_i whose cardinality is such that $(|N_i|-1)/(n-1) \approx \rho$, we subtract 1 because we do not consider the agent i), for every k and n we have generated around 200 families with mean value of density in the interval $[0.05, 0.9]$. As an example in Figures 3.1 and 3.2 the results for sets of 25 evolution matrices of dimension 25×25 are presented.

In Figure 3.1 we observe that, as the mean density of the set increases, the number of matrices with second spectral radius equal 1 reduces and, for density greater or equal 0.2, almost all families have a second joint spectral radius strictly less than 1, which implies that in the corresponding dynamical system the agents always reach a consensus. This is more evident in Figure 3.2, in which the arithmetic means of the values of the second joint spectral radius are plotted in function of the mean density of the family.



(a) (b)

Figure 3.1: Families of 25 evolution matrices of dimension 25×25, where * plots the number of matrices with second spectral radius equal 1 and * plots the values of the second joint spectral radius of a specific family.

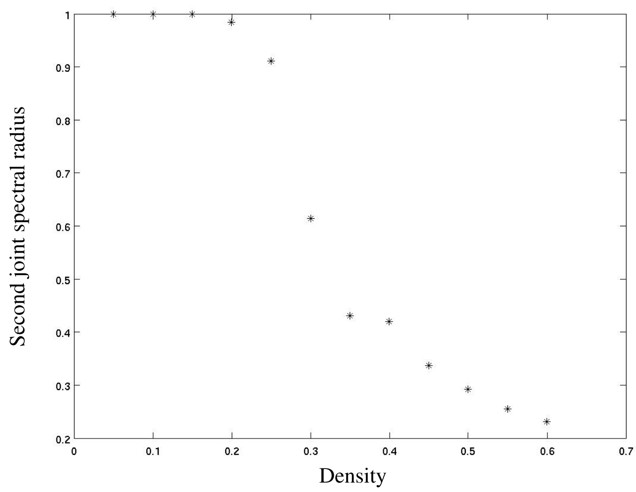


Figure 3.2: Average values of the second joint spectral radius for families of 25 matrices 25×25.

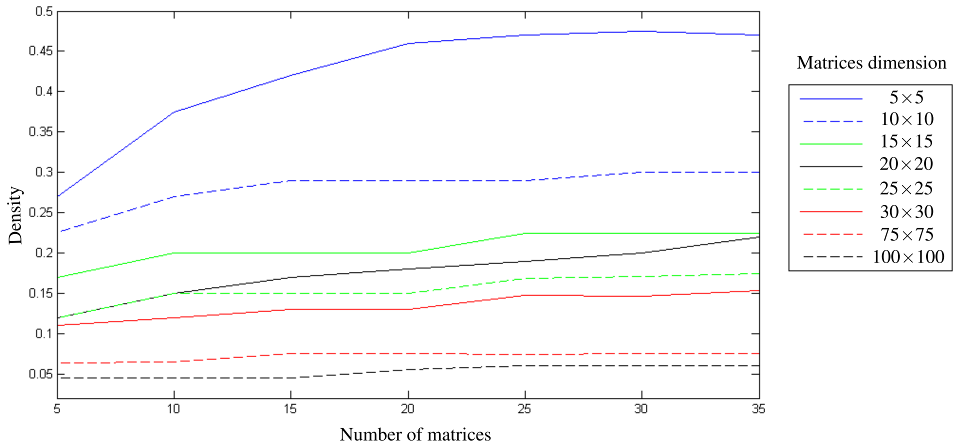


Figure 3.3: Density of the phase transition in function of the cardinality of $\mathscr{F}$ and the dimension of the matrices.

Consequently we are tempted to conclude that $\rho \approx 0.2$ is the value of density to which it corresponds a phase transition, from no net–transport to finite net–transport, in systems of 25 agents. However, if we repeat this study changing the cardinality k of the families, we discover that, as k increases, the phase transition tends to happen for increasing values of density. The same phenomenon occurs for every dimension n we have studied, as shown in Figure 3.3 (for $n = \{75, 100\}$ the curves have still a positive slope, even though almost imperceptible).

This is due to the fact that, as k increases, the probability that one of the matrices in the set presents two eigenvalues equal 1 in its spectrum rises. Consider, in fact, that the random matrices we are generating, matrices in which all the agents are supposed to have almost the same number of neighbors, has an effective mean density that is not exactly the one desired. Therefore the more we increase the cardinality of $\mathscr{F}$ the more it rises the probability that we pick a matrix with a mean density lower than the one desired and, thus, such that it has two eigenvalues equal 1 (low density implies high probability to have two or more disconnected subsets of agents). The same reasoning explains also the fact that the more we increase the dimension n of the matrices the slower the density of the phase transition rises (the bigger is n the more the effective mean value of the density is close to the desired value and, so, the probability that at least one matrix in $\mathscr{F}$ has two or more disconnected subsets of agents does not increases substantially with k).

In this last Figure we observe also that, increasing the dimension n, we have a reduction in the values of the density of the phase transition. The phenomenon might be related to the fact that, even though the indices of the agents belonging to every neighborhood should be *uniformly distributed*[1], in practice, for matrices of small dimension, this distribution is not uniform. The bigger the dimension is the more the distribution tends to be uniform and less biased.

In summary we can say that in a system of autonomous agents, which can be modeled by the Vicsek protocol, the density associated with the phase transition should be the one measured for sets of evolution matrices with a large cardinality k and big dimension n.

In Appendix B we present the analysis of the second joint spectral radius of a family made up of 35 matrices of dimension 20 and with mean density around 0.249. We compute upper and lower bounds for the second joint spectral radius by means of the Gripenberg's algorithm and we use another algorithm, proposed by Protasov et al. [57], to evaluate an approximated extremal ellipsoidal norm for the family. In this way we can give good estimates of the convergence of the system to a global consensus in the worst possible configuration. We also compare these theoretical results with the outcome of a few experimentations obtained considering a particular initial configuration of the system $\boldsymbol{\theta}(0)$ and random products of evolution matrices. The effective rate of convergence of the system to a global consensus proves to be well estimated by the second joint spectral radius.

About the PageRank analysis in Appendix B we report also an example of a family $\mathscr{F}$ of 10 evolution matrices of dimension 125 and mean density around 0.074. In this case we discover that the candidate spectrum–maximizing product P is, indeed, just one of the matrix in the set (it is known that the length of the s.m.p. tends to be short for random families). This matrix P has a second eigenvalue that is close to 1 and a correspond-

[1] `http://www.mathworks.com/help/techdoc/ref/rand.html`

ing PageRank with a very particular feature: one agent is the leader of all the network (this happen when the agent is by itself a minimal stable set and all the other agents are connected to it by a directed path). As long as the evolution of the system is represented just by the matrix P, we witness a very slow convergence of $\theta(t)$, solution of equation (3.12), to the vector $e\theta_{cons}$, where θ_{cons} is a scalar equal to the initial heading $\theta_l(0)$ of the leader (being l his corresponding index); slow convergence explained by the presence of a second spectral radius almost equal 1 in the spectrum of P. Nevertheless, since using the Gripenberg's algorithm we are able to find an upper bound of the second joint spectral radius which is strictly less then 1, we are guaranteed that the system will tend for $t \to \infty$ always to a global consensus. Furthermore, since a generic product of evolution matrices in the set $R = F_{j_m} \cdot \ldots \cdot F_{j_2} \cdot F_{j_1}$, with indices $j_i \in \{1, \ldots, k\}$, has usually a small second spectral radius, after a few iterations the system reach approximately a global consensus $e\widehat{\theta}_{cons}$, with $\widehat{\theta}_{cons}$ given by the inner product between the PageRank of the matrix R and the vector of the initial configuration $\theta(0)$.

Choosing a particular initial configuration $\theta(0)$ and analyzing how the system evolves for different choices of products of evolution matrices, we can observe a classical *butterfly effect*: small changes in the first steps, either in the initial configuration $\theta(0)$ or in the chosen sequence of evolution matrices, can determine substantial differences in the final configuration of the system.

3.3 Global consensus in networks with broken links

In this section we give alternative proofs to a few Theorems, presented by Di Cairano et al. in [44], valid for componentwise weakly connected digraph with fixed topology.

We start recalling the evolution equation of the Vicsek model

$$\theta(t+1) = F_{i_t}\theta(t)$$

where $\theta(t) = [\theta_1(t), \ldots, \theta_n(t)]^T$, $F_{i_t} = (D(t)+I)^{-1}(A(t)+I)$, I the identity matrix and $D(t)$ the degree matrix associated with the dynamic weighted digraph $\mathscr{G}(t) = (\mathscr{V}, \mathscr{E}(t), A(t))$.

This equation, obtained considering a discrete time step of size $\Delta t = 1$, can be interpreted, in the case of fixed topology $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$, as a discretization of a continuous–time relation.

Recalling from Definition 3.1.7 that the graph Laplacian L is given by $L = D - A$, we can rewrite

$$F = (D+I)^{-1}(A-D+D+I) = \\ (D+I)^{-1}(A-D)+I = I-(D+I)^{-1}L \tag{3.21}$$

therefore

$$\theta(t+1) = F\theta(t) = \left[I - K^{-1}L\right]\theta(t) = \left[I - \widetilde{L}\right]\theta(t) \tag{3.22}$$

where $K = D + I$ is a diagonal matrix and $\widetilde{L} = K^{-1}L$ is what we call the *generalized graph Laplacian*. Following Definition 3.1.8, we can associate with the generalized graph Laplacian $\widetilde{L}$ a *generalized Perron matrix* given by $\widetilde{P} = I - \Delta t\widetilde{L}$. We observe that the evolution matrix F is nothing more that the generalized Perron matrix associated with the digraph $\mathscr{G}$ when $\Delta t = 1$

$$\theta(t+1) = F\theta(t) = \widetilde{P}\theta(t). \tag{3.23}$$

In addition, considering a generic time step Δt, the discrete–time collective dynamics (3.23) of a network with fixed topology becomes

$$\theta(t+1) - \theta(t) = -\Delta t\widetilde{L}\theta(t) \tag{3.24}$$

which can be reformulated in continuous–time as

$$\dot{\theta} = -\widetilde{L}\theta. \tag{3.25}$$

The model associated with this generalized graph Laplacian, for any diagonal matrix K with positive diagonal elements, is said to be a *weighted-average consensus model* [46, Section II.F] to distinguish it from the *average consensus* model given by

$$\dot{\theta} = -L\theta. \tag{3.26}$$

This latter equation can be interpreted as a continuous–time version of the discrete–time relation

$$\theta(t+1) = (I-L)\theta(t) = P\theta(t) \tag{3.27}$$

where $P = I - L$ is the Perron matrix of $\mathscr{G}$ for a time–step $\Delta t = 1$ (ref Olfati–Saber et al. [46]).

Di Cairano et al. in [44] studied the case of average consensus models, therefore they considered continuous and discrete time models containing only matrix L and $P = I - \Delta t L$, but their results extend naturally to the case of generalized graph Laplacian $\widetilde{L} = K^{-1}L$ and the associated generalized Perron matrix $\widetilde{P} = I - \Delta t \widetilde{L}$ for every diagonal matrix K with positive diagonal elements, included the case previously studied of $K = D + I$.

We recall also the following Lemma which establishes a direct relation between the strong connectivity of a weighted simple digraph and the rank of its graph Laplacian.

3.3.1 Lemma ([46, Lemma 2])**.** *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ be a weighted simple digraph with graph Laplacian L defined in (3.5).*
If $\mathscr{G}$ is strongly connected, then
$\mathrm{rank}(L) = n - 1$ *and all nontrivial eigenvalues of L have positive real parts.*
If, instead $\mathscr{G}$ has $c > 1$ strongly connected components, then
$\mathrm{rank}(L) = n - c$.

According to this Lemma the Laplacian of a strongly connected weighted digraph has an isolated eigenvalue at zero, while a digraph with $c > 1$ strongly connected components has c eigenvalues at zero.

For strongly connected digraph the opposite implication does not hold in general. A counterexample is given by the weighted simple digraph with the following adjacency matrix A and graph Laplacian L

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{3.28}$$

Clearly $\mathrm{rank}(L) = n - 1 = 1$, but the digraph is not strongly connected since there is no path connecting v_2 to v_1.

As a Corollary of the previous Lemma we have that

3.3.2 Corollary. *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ be a weighted simple digraph with graph Laplacian L and evolution matrix F defined by (3.5) and (3.21), respectively. If $\mathscr{G}$ has $c > 1$ strongly connected components, then 1 is a semisimple eigenvalue of F and it has multiplicity c.*

Proof. First we observe that $\text{rank}(L) = n - c$ if and only if 0 is in the spectrum of L with multiplicity c. So, since F has exactly the same eigenvalues of $-(D+I)^{-1}L$ translated by 1 on the right and forasmuch as $(D+I)^{-1}L$ has in its spectrum exactly as many 0 eigenvalues as L, F has 1 as eigenvalue with multiplicity exactly c. Since F is row–stochastic, the semisimplicity of the eigenvalue 1 follows from Lemma 2.3.1(v). □

Given a weighted simple digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ we assume that there are m minimal stable sets $\mathscr{X}_i$ which we call *authorities*, for every $i = 1, \ldots, m$, and that correspond to m strongly connected components of $\mathscr{G}$, Lemma 3.1.4. We indicate by $S_a = \cup_{i=1}^{m} \mathscr{X}_i$ the set containing all the authorities of $\mathscr{G}$, while by S_0 we identify the remaining nodes i.e. $S_0 = \mathscr{V} \backslash S_a$. We assume the cardinality of S_0 and S_a to be n_0 and n_a, respectively.

It is easy to prove that

3.3.3 Lemma. *Given $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ and defined the set of vertices which do not belong to any authorities $S_0 = \mathscr{V} \backslash S_a$, S_0 does not contain any stable set. From any node $v_i \in S_0$ there exists a path to S_a.*

Let us call L_i the graph Laplacian associated with the subdigraph induced by each minimal stable set $\mathscr{X}_i \subseteq \mathscr{V}$ on $\mathscr{G}$, which we suppose to be m. The graph Laplacian of $\mathscr{G}$, after a suitable reordering of the vertices in V, can be written as

$$L = \begin{bmatrix} L_a & 0 \\ R & F \end{bmatrix}, \quad \text{where} \quad L_a = \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_m \end{bmatrix} \tag{3.29}$$

The matrix $R = [r_{ij}]_{n_0 \times n_a}$ contains the edges from nodes in S_0 to nodes in S_a, while

$$F = L_\Psi + H,$$

where L_Ψ is the Laplacian of the subdigraph Ψ induced by S_0 on $\mathscr{G}$ and H is a diagonal matrix that compensate for the rows of R in the Laplacian L, i.e. $h_{ii} = -\sum_{j=1}^{n_a} r_{ij} \geq 0$.

3.3.4 Lemma ([44, Lemma 4])**.** *Consider a weighted simple digraph Ψ on k vertices with graph Laplacian L_Ψ and a nonnegative diagonal matrix $H \in \mathbb{R}^{k\times k}$, if we define $M = L_\Psi + H$, then* $\det M \geq 0$.

Proof. Since $M \in \mathbb{R}^{k\times k}$ its eigenvalues can be only real or complex conjugate numbers so that $\det M \in \mathbb{R}$. Given the definition of graph Laplacian, the center of every Geršgorin circle associated with L_Ψ lies on $\mathbb{R}^+$ and all these circles pass through the origin [24, Geršgorin discs 6.1]. The i–th Geršgorin circle of $M = L_\Psi + H$ is the same of L_Ψ, but with the center shifted on the right of $h_{ii} \geq 0$. Then, every eigenvalue of M has a nonnegative real part, hence, $\det M \geq 0$. □

For the proof of the following Lemma we need a Corollary of the Geršgorin Theorem

3.3.5 Proposition ([24, Corollary 6.2.27])**.** *If $F = [f_{ij}]_{i,j=1}^k$ is irreducible, diagonally dominant, i.e. $|f_{ii}| \geq \sum_{j\neq i} |f_{ij}|$ for all $i = 1,\dots,k$, and for at least one value $i = 1,\dots,k$ we have $|f_{ii}| > \sum_{j\neq i} |f_{ij}|$, then*
F is invertible, i.e. 0 *is not an eigenvalue of F.*

3.3.6 Lemma ([44, Lemma 5])**.** *Consider the assumptions of Lemma 3.3.4 and assume that for every minimal stable set $\mathscr{Y}_i$ on Ψ there exists at least one node $v_j \in \mathscr{Y}_i$ such that $h_{jj} > 0$. Then* $\det M > 0$.

Proof. It is known that for every matrix $M \in \mathbb{R}^{k\times k}$ there exists a permutation matrix $P \in \mathbb{R}^{k\times k}$ such that

$$N = PMP^T = \begin{bmatrix} M_{11} & 0 & \cdots & 0 \\ M_{21} & M_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ M_{r1} & M_{r2} & \cdots & M_{rr} \end{bmatrix} \tag{3.30}$$

where each block M_{ii}, for $i = 1,\dots,r$, is either an irreducible square matrix or a 1×1 null matrix. If $r = 1$ the original matrix M is irreducible [35].

Since $M \in \mathbb{R}^{k\times k}$ satisfies all the hypotheses of the previous Lemma, it follows that $\det M = \det N = \prod_{i=1}^r \det M_{ii} \geq 0$. Each matrix M_{ii} is irreducible and, in the case in analysis, diagonally dominant, so, it corresponds to a subset of vertices $\mathscr{Y}_i$ of the digraph Ψ which are strongly connected each other. $\mathscr{Y}_i$, in general, is not a minimal stable set on Ψ, but only a minimal stable set on the subdigraph induced by $\mathscr{Y}_i$ on Ψ. This implies that, for each $i = 1,\dots,r$, we can write $M_{ii} = L_{\mathscr{Y}_i} + H_i + D_i$ where $L_{\mathscr{Y}_i}$ is the Laplacian of the strongly connected weighted subdigraph induced by the subset of vertices $\mathscr{Y}_i$ on Ψ, D_i is a diagonal matrix whose entry $D_i[h,h]$ is equal to the sum of the sign changed elements in the h–row of all the matrices M_{ij}, with $j < i$, and H_i is the diagonal matrix given by the elements of H corresponding to the vertices in $\mathscr{Y}_i$. If for a certain $i \in \{1,\dots,r\}$ all the matrices $M_{ij} = 0$, with $j < i$, then the corresponding $\mathscr{Y}_i$ is a minimal stable set also on Ψ and M_{ii} is just $M_{ii} = L_{\mathscr{Y}_i} + H_i$. In this last case, by hypothesis, there exists at least one node $v_j \in \mathscr{Y}_i$ such that $H_i[j,j] > 0$. So for every M_{ii}, $i = 1,\dots,r$, at least one matrix among H_i and D_i is not zero, therefore the hypotheses of Proposition 3.3.5 hold and $\det M_{ii} > 0$, for every $i = 1,\dots,r$. For this reason $\det M > 0$. □

We observe that Lemmas 3.3.4 and 3.3.6 hold true also if we consider a generalized graph Laplacian $\widetilde{L}_\Psi = K^{-1}L_\Psi$ instead of the graph Laplacian L_Ψ, with K any possible diagonal matrix with positive diagonal elements.

3.3.7 Theorem ([44, Theorem 1]). *For any componentwise weakly connected digraph $\mathscr{G} = (\mathscr{V},\mathscr{E},A)$, with graph Laplacian L, and for fixed agreement values $\mu_1,\dots,\mu_m$ of the authorities $\mathscr{X}_1,\dots,\mathscr{X}_m$ of $\mathscr{G}$, there always exists a single equilibrium point $\theta = [\mu_1,\dots,\mu_1,\mu_2,\dots,\mu_2,\dots,\mu_m,\dots,\mu_m,\theta_{n_a+1},\dots,\theta_{n_a+n_0}]^T$ such that $L\theta = 0$, where $S_0 = \mathscr{V}\backslash S_a$ is the set of agents which do not belong to any authorities and n_0, n_a are the cardinality of S_0 and S_a, respectively.*

Proof. We look for the solutions θ of $L\theta = 0$. If we consider the general form of the graph Laplacian given in (3.29), where $L_1,\dots,L_m$ are the graph Laplacians of the strongly connected subdigraphs induced by the m authorities $\mathscr{X}_i$ on $\mathscr{G}$, we can rewrite $L\theta = 0$ as

$$L\theta = \begin{bmatrix} L_a & 0 \\ R & F \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = 0 \tag{3.31}$$

where $w = [\theta_1,\dots,\theta_{n_a}]$ and $z = [\theta_{n_a+1},\dots,\theta_{n_a+n_0}]$.

Since we know that each L_i has an isolated eigenvalue at zero with a corresponding eigenvector of all equal entries (ref Lemma 3.3.1), we have that, for every $i=1,\dots,m$, the unique solution of $L_i w_i = 0$ is given by $w_i = \mu_i e$ with μ_i which depends only on the initial state of the vertices in $\mathscr{X}_i$ (ref [46, Theorem 1]). The first n_a rows of the system (3.31) have, thus, the unique solution

$$L_a w = \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_m \end{bmatrix} \begin{bmatrix} \mu_1 e_{(1)} \\ \mu_2 e_{(2)} \\ \vdots \\ \mu_m e_{(m)} \end{bmatrix} = 0 \tag{3.32}$$

with $e_{(i)}$, $i=1,\dots,m$, column vector of all ones with a suitable length. The last n_0 rows in (3.31) can be rewritten in function of the unique solution w of (3.32)

$$Fz = -Rw. \tag{3.33}$$

This equation admits a unique solution if and only if F is nonsingular (invertible).

For the nonsingularity of F we can use Lemma 3.3.6. Let Ψ be the digraph induced by S_0 on $\mathscr{G}$, then we can write $F = L_\Psi + H$ as explained on page 91. There exists always a permutation matrix P such that

$$B = \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} L_a & 0 \\ R & F \end{bmatrix} \begin{bmatrix} I \\ P^T \end{bmatrix} = \begin{bmatrix} L_a & 0 \\ \widehat{R} & \widehat{F} \end{bmatrix} \tag{3.34}$$

$$\widehat{F} = PFP^T = \begin{bmatrix} \widehat{F}_{11} & 0 & \cdots & 0 \\ \widehat{F}_{21} & \widehat{F}_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \widehat{F}_{r1} & \widehat{F}_{r2} & \cdots & \widehat{F}_{rr} \end{bmatrix}, \quad \widehat{R} = PR = \begin{bmatrix} \widehat{R}_1 \\ \vdots \\ \widehat{R}_r \end{bmatrix} \tag{3.35}$$

with $\widehat{F}_{ii}$ irreducible and diagonally dominant by construction, for every $i=1,\dots,r$. We can write $\widehat{F}_{ii} = L_{\mathscr{Y}_i} + H_i + D_i$ where $L_{\mathscr{Y}_i}$ is the Laplacian of the strongly connected weighted subdigraph induced by the subset of vertices $\mathscr{Y}_i$ on Ψ, D_i is a diagonal matrix whose entry $D_i[h,h]$ is equal to the sum of the sign changed elements in the h–row of all the matrices $\widehat{F}_{ij}$, with $j<i$,

and H_i is the diagonal matrix given by the elements of H corresponding to the vertices in $\mathscr{Y}_i$. Since for every $\mathscr{Y}_i$ corresponding to a minimal stable set on Ψ, but not on $\mathscr{G}$, there exists at least one vertex $v_j \in \mathscr{Y}_i$ that has an edge with a vertex in S_a, i.e. $H_i[jj] = -\sum_{k=1}^{n_a} \widehat{R}_i[j,k] > 0$, then the hypotheses of Lemma 3.3.6 hold, so $\det \widehat{F} = \det F > 0$ and the matrix F is invertible.

In conclusion, the unique solution of (3.31) is given by

$$z = -F^{-1}Rw. \tag{3.36}$$

where w is the unique solution of (3.32). □

3.3.8 Corollary ([44, Corollary 1]). *If the assumptions of Theorem 3.3.7 hold and $\mu_1 = \ldots = \mu_m = \mu$, the unique solution of $L\theta = 0$ is the global consensus $\theta = \mu e$.*

Proof. If we consider the last n_0 rows of matrix L in equation (3.31) we have the submatrix $[R\,F]$ which is such that $[R\,F]\,\mu e = 0$, since every row of the graph Laplacian adds up to zero. By hypothesis $w = \mu e$, so we have that $[w\,z]^T = \mu e^T$ is a solution of the system and it is also the unique one by Theorem 3.3.7. □

The solution of $L\theta = 0$, given by the previous Theorem, is also the limit solution of both equation (3.26) and (3.27). Furthermore Theorem 3.3.7 still holds true if we consider, instead of the graph Laplacian L, the generalized graph Laplacian $\widetilde{L} = K^{-1}L$ associated with the digraph $\mathscr{G}$, where K is any diagonal matrix with positive diagonal elements. So, even for $\widetilde{L}\theta = 0$, we have a unique solution θ which is also the limit solution of both equation (3.25) and (3.23).

Finally, making use of the following Lemma, we prove a Theorem which guarantees a monotonicity on the consensus values reached by the nodes in S_0 with respect to the consensus attained by the authorities: if the differences in the consensuses values reached by the authorities are small, the values chosen in the limit by the agents belonging to S_0 are close to each other, and bounded by the ones of the authorities.

3.3.9 Lemma ([44, Corollary 3]). *Considering the digraph of Theorem 3.3.7, equation (3.36) written in the form $\bar{z} = -F^{-1}R\bar{w} = \Phi\bar{w}$ and given*

the two vectors $\bar{w}^{(1)}$ and $\bar{w}^{(2)}$, if $\bar{w}^{(1)} \geq \bar{w}^{(2)}$, then the corresponding solutions satisfy $\bar{z}^{(1)} \geq \bar{z}^{(2)}$.
Furthermore, for every agent v_i in S_0, $i = 1, \dots, n_0$, $\bar{\theta}_i$ is the convex combination of $\mu_1 \leq \dots \leq \mu_m$, which are the consensus values of the authorities $\mathscr{X}_1, \dots, \mathscr{X}_m$.

Proof. As stated in the proof of Corollary 3.3.8

$$Le = \begin{bmatrix} L_a & 0 \\ R & F \end{bmatrix} e = 0 \tag{3.37}$$

and $\Phi = -F^{-1}R$ is such that $\sum_{j=1}^{n_a} \phi_{ij} = 1$ for all $i = 1, \dots, n_0$.

We prove, in addition, that $\Phi \geq 0$. This implies that every row of Φ can be interpreted as a sequence of weights of a convex sum.

$\Phi = F^{-1}(-R)$ with $-R \geq 0$ by construction. We define Λ as a diagonal matrix such that $\lambda_{ii} = f_{ii}$ for all $i = 1, \dots, n_0$. This matrix is invertible since for every node $v_i \in S_0$, $i = 1, \dots, n_0$, it results that $f_{ii} > 0$ (if the vertex v_i is a *sink* of Ψ, i.e. it has no edges with other vertices in S_0, there must exist at least one entry $r_{ij} < 0$, for some $j = 1, \dots, n_a$, such that $f_{ii} = -r_{ij} > 0$).

$$F^{-1} = (\Lambda - N)^{-1} = \left(\Lambda\left(I - \Lambda^{-1}N\right)\right)^{-1} = \left(I - \Lambda^{-1}N\right)^{-1}\Lambda^{-1} \tag{3.38}$$

$$\Lambda^{-1} = \begin{bmatrix} 1/\lambda_{11} & & 0 \\ & \ddots & \\ 0 & & 1/\lambda_{n_0 n_0} \end{bmatrix} \geq 0 \tag{3.39}$$

Given $\lambda_{ii} \geq \sum_{j \neq i} |n_{ij}|$ it follows that $\sum_{j \neq i} \frac{|n_{ij}|}{\lambda_{ii}} \leq 1$, so matrix $I - \Lambda^{-1}N$ is diagonally dominant and invertible. The invertibility can be proved following the same reasoning as in the proof of Lemma 3.3.6.

Furthermore we can write

$$\left(I - \Lambda^{-1}N\right)^{-1} = \sum_{j=0}^{\infty} \left(\Lambda^{-1}N\right)^j, \text{ with } \left(\Lambda^{-1}N\right)^j \geq 0 \quad \forall j = 0, 1, \dots \tag{3.40}$$

which implies that also the sum is nonnegative, then $F^{-1} \geq 0$ and hence $\Phi \geq 0$.

Given that

$$\bar{z}^{(r)} = \Phi \bar{w}^{(r)} \text{ for } r = 1, 2 \tag{3.41}$$

if $\bar{w}^{(1)} \geq \bar{w}^{(2)}$, by linearity of Φ and its nonnegative, we get the nondecreaseness, i.e. $\bar{z}^{(1)} \geq \bar{z}^{(2)}$.

Finally, supposed

$$\bar{z} = -F^{-1}R\bar{w} = \Phi \begin{bmatrix} \mu_1 e_{(1)} \\ \mu_2 e_{(2)} \\ \vdots \\ \mu_m e_{(m)} \end{bmatrix}. \tag{3.42}$$

For every $i = 1, \dots, n_0$, we have that

$$\bar{z}_i = \sum_{j=1}^{n_a} \phi_{ij} \bar{w}_j = \sum_{j=1}^{n_{a_1}} \phi_{ij} \mu_1 + \dots + \sum_{j=n_{a_{h-1}}+1}^{n_{a_h}} \phi_{ij} \mu_h + \dots + \sum_{j=n_{a_{m-1}}+1}^{n_{a_m}} \phi_{ij} \mu_m \tag{3.43}$$

where n_{a_h} is the cardinality of the authority $\mathscr{X}_h$, for $h = 1, \dots, m$. Thus $\bar{z}_i$ is equal to the convex sum of $\mu_1 \leq \dots \leq \mu_m$. □

3.3.10 Theorem ([44, Theorem 3]). *Assume $\mu_1 \leq \dots \leq \mu_m$ be the consensus values of the authorities $\mathscr{X}_1, \dots, \mathscr{X}_m$. Then $\mu_1 \leq \bar{z}_i \leq \mu_m$, for every $i = 1, \dots, n_0$.*

Proof. It follows immediately from previous Lemma. In fact $\bar{z}_i$ is equal to a convex sum of the values $\mu_1 \leq \dots \leq \mu_m$, for every $i = 1, \dots, n_0$. □

3.4 Conclusions

The joint spectral radius proves to be an useful tool for analysing the convergence to a global consensus in a systems of agents and in particular the robustness of this convergence with respect to changes over time in the topology of the system. These changes can be related to the agents behavior or to external events like a failure in communications or a malicious attack. We presented experimental results, a detailed analysis of a case of study and we discussed the connections with the Google model.

Inspired by this connection we plan to study in a future work the PageRank from a dynamical point of view. In particular we are interested in analysing the influence of an agent, or group of agents, in the final value θ_{cons} in a dynamical context. This study may help to shed new light on complex processes such as decision making and information diffusion, which can be found in many contexts like in the study of infectious diseases diffusion, in viral advertising, in economics, in finance and in many social sciences.

Part III

Generic families

Chapter 4

Theory

This Chapter, and more in general this third part of the thesis, deals with the study of spectral properties of families of square matrices.

Last two decades have been characterized by an increasing interest in the analysis of the maximal growth rate of long products generated by matrices belonging to a specific set/family. The maximal growth rate can be evaluated considering a generalization of the spectral radius of a single matrix to the case of a set of matrices.

This generalization can be formulated in many different ways, nevertheless in the commonly studied cases of bounded or finite families all the possible generalizations coincide in a unique value that is usually called *joint spectral radius* or simply *spectral radius*. The joint spectral radius, however, can prove to be hard to compute and can lead even to undecidable problems. We present in this Chapter all the possible generalizations of the spectral radius, their properties and the associated theoretical challenges.

From an historical point of view the first two generalizations of spectral radius, the so–called joint and common spectral radius, were introduced by Rota and Strang in the three pages paper "*A note on the joint spectral radius*" published in 1960 [110]. After that more than thirty years had to pass before a second paper was issued on this topic: in 1992 Daubechies and Lagarias [70] published "*Sets of matrices all infinite products of which converge*" introducing the generalized spectral radius, conjecturing it was equal to the joint spectral radius (this was proven immediately after by

Berger and Wang [52]) and presenting examples of applications. From then on there has been a rapidly increasing interest on this subject and the more years pass the more the number of mathematical branches and applications directly involved in the study of these quantities increases [53].

The study of infinite products convergence properties proves to be of primary interest in a variety of contexts:
Nonhomogeneous Markov chains, deterministic construction of functions and curves with self-similarities under changes in scale like the von Koch snowflake and the de Rham curves, two-scale refinement equations that arise in the construction of wavelets of compact support and in the dyadic interpolation schemes of Deslauriers and Dubuc [70, 109], the asymptotic behavior of the solutions of linear difference equations with variable coefficients [77, 78, 79], coordination of autonomous agents [89, 103, 73], hybrid systems with applications that range from intelligent traffic systems to industrial process control [61], the stability analysis of dynamical systems of autonomous differential equations [65], computer–aided geometric design in constructing parametrized curves and surfaces by subdivision or refinement algorithms [101, 66], the stability of asynchronous processes in control theory [115], the stability of desynchronised systems [93], the analysis of magnetic recording systems and in particular the study of the capacity of codes submitted to forbidden differences constraints [102, 56], probabilistic automata [105], the distribution of random power series and the asymptotic behavior of the Euler partition function [109], the logarithm of the joint spectral radius appears also in the context of discrete linear inclusions as the Lyapunov indicator [51, 84]. For a more extensive and detailed list of applications we refer the reader to the Gilbert Strang's paper "*The Joint Spectral Radius*" [113] and to the doctoral theses by Jungers and Theys [90, 114].

The Chapter develops as following: in Section 4.1 we add notation and terminology to those presented in Section 1.1; Section 4.2 presents first a case of study associated with the asymptotic behavior analysis of the solutions of linear difference equations with variable coefficients, further, it contains the definitions and properties of all the possible generalizations of spectral radius for a set of matrices, in particular the irreducibility, nondefectivity and finiteness properties are discussed.

4.1 Terminology, notation and basic properties

In this Section we add notation, terminology, definitions and properties, to those presented in Section 1.1, which are employed in this third part of the thesis.

We use the expression $\mathbb{N}_0$ meaning the set of natural numbers, included zero.

A matrix B is said to be *normal* if $BB^* = B^*B$, *unitary* if $BB^* = B^*B = I$, *Hermitian* if $B = B^*$. Hermitian and unitary matrices are, by definition, normal matrices.

A *proper subset* of a set $\mathscr{A}$ is a set $\mathscr{B}$ that is strictly contained in $\mathscr{A}$. This is written as $\mathscr{B} \subsetneq \mathscr{A}$.

In Section 1.1 we presented the Jordan canonical form, now we introduce an additional matrix factorization, the so–called *singular value decomposition* (in short svd): Given a square matrix $A \in \mathbb{C}^{n\times n}$ with rank $k \leq n$, there always exists a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ with nonnegative diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > \sigma_{k+1} = \cdots = \sigma_n = 0$ and two unitary matrices $U, V \in \mathbb{C}^{n\times n}$ such that $A = U\Lambda V^*$, which is defined as the singular value decomposition of A. The matrix $\Lambda = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ is always uniquely determined and $\sigma_1^2 \geq \cdots \geq \sigma_n^2$ correspond to the eigenvalues of the Hermitian matrix AA^*. Values $\sigma_1, \ldots, \sigma_n$ are the so–called *singular values* of A.

The *trace* of an $n\times n$–matrix A, denoted by $\mathrm{tr}(A)$, is given by the sum of the diagonal elements of A, $\mathrm{tr}(A) = \sum_{i=0}^{n} a_{ii}$, and it is also equal to the sum of all the eigenvalues in the spectrum of A, $\mathrm{tr}(A) = \sum\limits_{\lambda\in\sigma(A)} \lambda$.

The *spectral radius* of a square matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\} \tag{4.1}$$

It is easy to prove that $\rho(A^k) = (\rho(A))^k$ for every $k \in \mathbb{N}$ and, thus, given a generic power k of the matrix A, the value $(\rho(A^k))^{1/k}$ is just equal to the spectral radius of the matrix.

It is possible to characterize the spectral radius using the trace of the matrix. Since $\lambda^k \in \sigma(A^k)$ for every eigenvalue $\lambda \in \sigma(A)$ and for every $k \in \mathbb{N}$, it follows that

$$|\mathrm{tr}\,(A^k)|^{1/k} = \rho(A) \left| \sum_{\lambda \in \sigma(A)} \lambda^k / (\rho(A))^k \right|^{1/k} \tag{4.2}$$

which converges to $\rho(A)$ as $k \to \infty$

$$\rho(A) = \lim_{k \to \infty} \left| \mathrm{tr}\,(A^k) \right|^{1/k}. \tag{4.3}$$

For a square matrix A and for $p \in [1, \infty]$, $\|A\|_p$ is the matrix norm induced by the corresponding p–vector norm (ref Section 1.1). The induced matrix norms are sometimes defined as *operator norms* [24, Definition 5.6.3]. Among the induced matrix norms we will make use of the following

The maximum column–sum matrix norm $- \; p = 1$

$$\|A\|_1 = \max_{\|x\|_1 = 1} \|Ax\|_1 = \max_j \sum_i |a_{i,j}|$$

The spectral norm $- \; p = 2$

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 = \sigma_1(A) = \sqrt{\rho(AA^*)}$$

The maximum row–sum matrix norm $- \; p = \infty$

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty = \max_i \sum_j |a_{i,j}|$$

Every induced matrix norm $\|\cdot\|_*$ is submultiplicative i.e. $\|AB\|_* \leq \|A\|_* \|B\|_*$ for every square matrix A and B.

Another family of induced matrix norms are the *ellipsoidal norms.* Let us consider an Hermitian positive definite matrix $P \succ 0$ (i.e. P is a nonsingular Hermitian matrix such that $x^* P x > 0$ for all nonzero $x \in \mathbb{C}^n$ or, equivalently, P is a Hermitian matrix such that all its eigenvalues are strictly positive). The *vector ellipsoidal norm* is defined as

$$\|x\|_P = \sqrt{x^* P x}. \tag{4.4}$$

The corresponding induced matrix norm is given by

$$\|A\|_P = \max_{\|x\|_P = 1} \|Ax\|_P = \max_{\sqrt{x^* P x} = 1} \sqrt{x^* A^* P A x} \tag{4.5}$$

Recalling that [24, Corollary 7.2.9] P is positive definite if and only if there exists a nonsingular upper triangular matrix $T \in \mathbb{C}^{n\times n}$, with strictly positive diagonal entries, such that $P = T^*T$, which is defined as the *Cholesky decomposition* of P, we can rewrite

$$\|A\|_P = \max_{\sqrt{x^*T^*Tx}=1} \sqrt{x^*A^*T^*TAx} = \max_{\|Tx\|_2=1} \|TAx\|_2$$

and if we rename $y = Tx$, T by construction is nonsingular so $x = T^{-1}y$, we get

$$\|A\|_P = \max_{\|y\|_2=1} \|TAT^{-1}y\|_2 = \|TAT^{-1}\|_2 = \sqrt{\rho(TAT^{-1}(TAT^{-1})^*)} \tag{4.6}$$

Since T is nonsingular and remembering that the spectrum of a matrix is invariant under *similarity transformation*, two matrices M and $T^{-1}MT$ have the same eigenvalues, counting multiplicity. So from (4.6) we obtain that

$$\|A\|_P = \sqrt{\rho(TAP^{-1}A^*T^*)} = \sqrt{\rho(AP^{-1}A^*P)} \tag{4.7}$$

Given a generic power k of the matrix A, the value $\|A^k\|^{1/k}$ is defined as the *normalized norm* of the matrix, in the sense that is normalized with respect to the length of the product.

Given the family $\mathscr{F} = \{A_i\}_{i\in\mathscr{I}}$ of complex square $n\times n$–matrices, with $\mathscr{I}$ a set of indices, $\mathscr{F}$ is defined *bounded* if it does exist a constant $C < +\infty$ such that $\sup_{i\in\mathscr{I}} \|A_i\| \le C$. While we define the set *finite* if it is constituted by a finite number of matrices. Trivially a finite set is always bounded.

A matrix A is said to be *nondefective* if and only if its Jordan canonical form is diagonal i.e. each eigenvalue of A is semisimple or, equivalently, it has geometric multiplicity equal to algebraic multiplicity, otherwise A is defined *defective*. In this part of the thesis we deal with a weaker condition of nondefectivity: a matrix A is said to be *weakly nondefective* if and only if the eigenvalues of A with modulus equal to the spectral radius, i.e. with maximal modulus, are semisimple, if it is not the case the matrix is defined *weakly defective*. Using the Jordan canonical form of A it is easy to prove that, whenever $\rho(A) > 0$, defined $A^* = A/\rho(A)$, A is weakly nondefective if and only if powers $(A^*)^k$ are bounded for every $k \ge 1$.

From now on, for the sake of simplicity and to be coherent with the literature on the spectral radius of sets, we use the expressions *strongly* *nondefective* and *strongly defective* in place of nondefective and defective, whereas we make use of the words *nondefective* and *defective* meaning weakly nondefective and weakly defective.

Let us now recall basic relations between spectral radius and matrix norms:

4.1.1 Theorem ([24, Theorem 5.6.9]). *If $\|\cdot\|$ is any matrix norm on $\mathbb{C}^{n\times n}$ and if $A \in \mathbb{C}^{n\times n}$, then*

$$\rho(A) \leq \|A\|.$$

Furthermore

4.1.2 Lemma ([24, Lemma 5.6.10]). *Let $A \in \mathbb{C}^{n\times n}$, for every $\varepsilon > 0$ there is a matrix norm $\|\cdot\|_\varepsilon$ such that*

$$\rho(A) \leq \|A\|_\varepsilon \leq \rho(A) + \varepsilon \tag{4.8}$$

The spectral radius of A is not itself a matrix or vector norm, but if we let $\varepsilon \to 0$ in (4.8) we have that $\rho(A)$ is the greatest lower bound for the values of all matrix norms of A

$$\rho(A) = \inf_{\|\cdot\| \in \mathscr{N}} \|A\| \tag{4.9}$$

where $\mathscr{N}$ denotes the set of all possible induced matrix norms (the so–called operator norms).

Spectral radius allows to characterize *convergent matrices*, i.e. those matrices whose successive powers tends to zero:

4.1.3 Theorem ([24, Theorem 5.6.12]). *Let $A \in \mathbb{C}^{n\times n}$, then*

$$\lim_{k\to\infty} A^k = 0 \qquad \Leftrightarrow \qquad \rho(A) < 1$$

As a Corollary of the previous Theorem we have the so–called *Gelfand's formula*:

4.1.4 Corollary ([24, Corollary 5.6.14]). *Let $\|\cdot\|$ be any matrix norm on $\mathbb{C}^{n\times n}$, then*

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k} \quad \textit{for all} \quad A \in \mathbb{C}^{n\times n} \tag{4.10}$$

The Gelfand's formula gives us two information:

- the spectral radius of A represent the asymptotic growth rate of the normalized norm of A^k: $\|A^k\|^{1/k} \sim \rho(A)$ as $k \to \infty$
- the normalized norm $\|A^k\|^{1/k}$ can be used to approximate the spectral radius and in the limit for $k \to \infty$ the two quantities coincide.

Given a row–stochastic matrix A its maximum row–sum matrix norm is equal 1 by definition of row–stochasticity (ref pages 5 and 73). By Theorem 4.1.1, choosing as matrix norm the maximum row–sum, we have that for every stochastic matrix A

$$\rho(A) \leq \|A\|_\infty = 1 \tag{4.11}$$

The row–stochasticity of A can be formulated also as

$$Ae = e \tag{4.12}$$

with e the all–ones vector and $\lambda = 1$ the eigenvalue of A associated with the right eigenvector $x = e$. So we have that $\rho(A) = 1$ (ref Lemma 2.3.1). Remembering that the set of stochastic matrices is closed under matrix multiplication, we observe that the very same result can be proved also using the Gelfand formula: choosing as matrix norm the maximum row–sum we have that $\|A^k\|_\infty^{1/k} = 1$ for every integer $k \geq 1$.

In the following we generalize all these notions to the case of a family of matrices.

4.2 Framework

4.2.1 A case of study

Given a stable discrete time system we want to analyze its robustness with respect to perturbations not a priori quantifiable.

Let us consider the system

$$x(k+1) = A_0 x(k), \qquad k \in \mathbb{N}_0. \tag{4.13}$$

with $x(0) \in \mathbb{C}^n$ and $A_0 \in \mathbb{C}^{n\times n}$ such that the system is asymptotically stable, i.e. $\rho(A_0) < 1$ (ref Theorem 4.1.3). We consider the perturbed system given by time–varying perturbations

$$x(k+1) = \left(A_0 + \sum_{i=1}^{p} \delta_i(k) A_i\right) x(k), \qquad k \in \mathbb{N}_0. \tag{4.14}$$

The matrices $\{A_i\}_{i=1}^p$ are known, but the perturbations $\{\delta_i(k)\}_{i=1}^p$ are not. The perturbations may depend on incomplete modeling, neglect of dynamics or measurement uncertainty. We are interested to know if a stability result for the theoretical model (4.13) holds also for the real system (4.14).

The perturbed system (4.14) can be regarded as a first order system of difference equations with variable coefficients

$$x(k+1) = Y_{i_k} x(k), \qquad k \in \mathbb{N}_0. \tag{4.15}$$

where $x(0) \in \mathbb{C}^n$ and $Y_{i_k} \in \mathbb{C}^{n\times n}$ is an element of the following family

$$\mathscr{F}_\alpha = \left\{ A_0 + \sum_{i=1}^{p} \delta_i A_i \;\middle|\; \|\delta\| \leq \alpha \right\} \tag{4.16}$$

where $\delta = (\delta_1 \;\; \delta_2 \;\; \cdots \;\; \delta_p)^T$ and the bound on the uncertainties is known. This kind of problems arise in several contexts such as when applying numerical methods to non–autonomous systems of differential equations.

From a point of view of robustness or worst case analysis the goal is to determine the largest uncertainty level α^* such that for every $\alpha < \alpha^*$ the system remains stable (see e.g. [118]).

If the sequence of matrices Y_{i_k} is known, for $k \geq 0$, then the solution of (4.15) is given by

$$x(k+1) = P_k x(0), \qquad \text{with } \; P_k = \prod_{j=1}^{k} Y_{i_j}, \; k \geq 1 \tag{4.17}$$

where asymptotic stability may be studied directly (although this is not an easy task in general). Nevertheless we want to study the case where the sequence $\{Y_{i_k}\}_{k\geq 1}$ is not known a priori and may be whatever.

4.2.1 Definition (Uniform asymptotic stability – u.a.s.)**.** *We say that* (4.15) *is* uniformly asymptotically stable *if*

$$\lim_{k\to\infty} x(k) = 0 \tag{4.18}$$

for any initial $x(0)$ *and any sequence* $\{Y_{i_k}\}_{k\geq 1}$ *of elements in* $\mathscr{F}_\alpha$.

It is easy to prove that Definition 4.2.1 is equivalent of requiring that any possible left product $Y_{i_k} \cdot Y_{i_{k-1}} \cdot \ldots \cdot Y_{i_1}$ of matrices from $\mathscr{F}_\alpha$ vanishes as $k \to \infty$.

We observe that in the context of the discrete linear inclusions some authors refer to the uniform asymptotic stability as *absolute asymptotic stability* [84, 114].

For the single matrix case we have that u.a.s. holds if and only if the spectral radius of the matrix is strictly less than one, while for the general case of a family of matrices $\mathscr{F}$ we are driven to the problem of computing the *joint spectral radius* of $\mathscr{F}$. The intrinsic difficulty in exploiting this quantity is due to the non–commutativity of matrix multiplication.

4.2.2 Definitions and properties

Definitions

From now on we consider always complex square $n\times n$–matrices and sub-multiplicative norms if not differently specified. Let $\mathscr{F} = \{A_i\}_{i\in\mathscr{I}}$ be a family of matrices, $\mathscr{I}$ being a set of indices.
For each $k = 1, 2, \ldots$, consider the set $\mathscr{P}_k(\mathscr{F})$ of all possible products of length k whose factors are elements of $\mathscr{F}$, that is

$$\mathscr{P}_k(\mathscr{F}) = \{A_{i_1} \cdot \ldots \cdot A_{i_k} \mid i_1, \ldots, i_k \in \mathscr{I}\}$$

and set

$$\mathscr{P}(\mathscr{F}) = \bigcup_{k\geq 1} \mathscr{P}_k(\mathscr{F}) \tag{4.19}$$

to be the *multiplicative semigroup* associated with $\mathscr{F}$. While, defined $\mathscr{P}_0(\mathscr{F}) := I$, we have

$$\mathscr{P}^*(\mathscr{F}) = \bigcup_{k\geq 0} \mathscr{P}_k(\mathscr{F}) \tag{4.20}$$

the *multiplicative monoid* associated with $\mathscr{F}$.

We present four different generalizations of the concept of spectral radius of a single matrix to the case of a family of matrices $\mathscr{F}$.

The first generalization is due to Rota and Strang, in the seminal paper [110] published in 1960 they presented the generalization of the notion of spectral radius as limit of the normalized norm of a single matrix:

4.2.2 Definition (Joint Spectral Radius – jsr)**.** *If* $\|\cdot\|$ *is any matrix norm on* $\mathbb{C}^{n\times n}$*, consider*

$$\widehat{\rho}_k(\mathscr{F}) := \sup_{P\in\mathscr{P}_k(\mathscr{F})} \|P\|^{1/k}, \qquad k\in\mathbb{N}$$

i.e. the supremum among the normalized norms of all products in $\mathscr{P}_k(\mathscr{F})$*, and define the* joint spectral radius *of* $\mathscr{F}$ *as*

$$jsr(\mathscr{F}) = \widehat{\rho}(\mathscr{F}) = \lim_{k\to\infty} \widehat{\rho}_k(\mathscr{F}) \tag{4.21}$$

The joint spectral radius does not depend on the matrix norm chosen thanks to the equivalence between matrix norms in finite dimensional spaces.

We observe that in the discrete linear inclusions literature the logarithm of the joint spectral radius is sometimes called Lyapunov indicator [51].

In 1992 Daubechies and Lagarias [70] introduced the generalized spectral radius as a generalization of the $\limsup$ over all the spectral radii $\rho(A^k)^{1/k}$, $k\geq 1$, which are, trivially, always equal to $\rho(A)$.

4.2.3 Definition (Generalized Spectral Radius – gsr)**.** *Let* $\overline{\rho}(\cdot)$ *denote the spectral radius of an* $n\times n$*–matrix, consider*

$$\overline{\rho}_k(\mathscr{F}) := \sup_{P\in\mathscr{P}_k(\mathscr{F})} \rho(P)^{1/k}, \qquad k\in\mathbb{N}$$

i.e. the supremum among the spectral radii of all products in $\mathscr{P}_k(\mathscr{F})$ *normalized taking their* k*–th root, and define the* generalized spectral radius *of* $\mathscr{F}$ *as*

$$gsr(\mathscr{F}) = \overline{\rho}(\mathscr{F}) = \limsup_{k\to\infty} \overline{\rho}_k(\mathscr{F}) \tag{4.22}$$

For this two definitions it has been proved by Daubechies and Lagarias [70, 71] the following

4.2.4 Proposition (Four members inequality)**.** *For any set of matrices* $\mathscr{F}$ *and any* $k\geq 1$

$$\overline{\rho}_k(\mathscr{F}) \leq \overline{\rho}(\mathscr{F}) = gsr(\mathscr{F}) \leq jsr(\mathscr{F}) = \widehat{\rho}(\mathscr{F}) \leq \widehat{\rho}_k(\mathscr{F}) \tag{4.23}$$

independently of the submultiplicative norm used to define $\widehat{\rho}_k(\mathscr{F})$*.*

As a consequence of this we have that:

$$\widehat{\rho}(\mathscr{F}) = \inf_{k\geq 1} \widehat{\rho}_k(\mathscr{F}) \tag{4.24}$$

$$\overline{\rho}(\mathscr{F}) = \sup_{k\geq 1} \overline{\rho}_k(\mathscr{F}) \tag{4.25}$$

For the first equality see also [90, Lemma 1.2]; for the second one, since $\rho(M^k) = \rho(M)^k$ for every $k \in \mathbb{N}$ and considering that by definition of $\limsup$

$$\limsup_{k\to\infty} \ \overline{\rho}_k(\mathscr{F}) = \inf_{k\geq 1} \ \sup_{n\geq k} \ \overline{\rho}_n(\mathscr{F}),$$

if it does exist a finite product $P \in \mathscr{P}_r(\mathscr{F})$, $r \in \mathbb{N}$, such that $\rho(P)^{1/r} = \overline{\rho}(\mathscr{F})$, then, for every $m \in \mathbb{N}$, $\rho(P^m)^{1/mr} = \overline{\rho}(\mathscr{F})$ and, thus, $\sup_{n\geq k} \overline{\rho}_n(\mathscr{F}) = \overline{\rho}(\mathscr{F})$ for every $k \in \mathbb{N}$. This last equality is valid also if it does not exists such a finite product, in fact in this case the $\sup$ is achieved only for $n \to \infty$. So in both cases it results $\inf_{k\geq 1} \sup_{n\geq k} \ \overline{\rho}_n(\mathscr{F}) = \sup_{k\geq 1} \ \overline{\rho}_k(\mathscr{F})$, i.e. equation (4.25) holds true.

A third definition has been introduced by Chen and Zhou in 2000 [67] and is based on a generalization of the formula associating the spectral radius of a matrix with its trace:

4.2.5 Definition (Mutual Spectral Radius – msr)**.** *Let* $\mathrm{tr}(P)$ *be the trace of the product* $P \in \mathscr{P}_k(\mathscr{F})$ *then* $\sup\limits_{P\in\mathscr{P}_k(\mathscr{F})} |\mathrm{tr}(P)|$ *is the maximal absolute value among all the traces of the products of length* k*. Define the* mutual spectral radius *of* $\mathscr{F}$ *as*

$$msr(\mathscr{F}) = \limsup_{k\to\infty} \sup_{P\in\mathscr{P}_k(\mathscr{F})} |\mathrm{tr}(P)|^{1/k} \tag{4.26}$$

We present now the last characterization of the spectral radius of a family of matrices. For bounded sets (ref Section 4.1) it is possible to generalize the concept, express in equation (4.9), of spectral radius as the $\inf$ over the set of all possible induced matrix norms of A.

4.2.6 Definition (Common Spectral Radius – csr)**.** *Given a norm* $\|\cdot\|$ *on the vector space* $\mathbb{C}^n$ *and the corresponding induced matrix norm, we define*

$$\|\mathscr{F}\| := \sup_{i\in\mathscr{I}} \|A_i\| \tag{4.27}$$

where we assume that the family $\mathscr{F}$ is bounded. We define the common spectral radius *of $\mathscr{F}$ (see [110] and [72]) as*

$$csr(\mathscr{F}) = \widetilde{\rho}(\mathscr{F}) = \inf_{\|\cdot\| \in \mathscr{N}} \|\mathscr{F}\| \tag{4.28}$$

where $\mathscr{N}$ denotes the set of all possible induced matrix norms.

This definition was first introduced by Rota and Strang in 1960 [110] and re–introduced 35 years later by Elsner [72].

In the case of bounded sets, it is possible to prove that the four characterizations we presented coincide.

4.2.7 Theorem (The Complete Spectral Radius Theorem)**.** *For a bounded family $\mathscr{F}$ the following equalities hold true*

$$gsr(\mathscr{F}) = jsr(\mathscr{F}) = csr(\mathscr{F}) = msr(\mathscr{F}) \tag{4.29}$$

The equality of gsr and jsr was conjectured by Daubechies and Lagarias and it was proven by Berger and Wang [52], Elsner [72], Chen and Zhou [67], Shih et al. [112]. For the equality of csr and jsr we refer the reader to the seminal work of Rota and Strang [110] or again [72]. Chen and Zhou [67] proved the last equality.

We observe that the first equality is the generalization of the Gelfand's formula (Corollary 4.1.4) to the case of a family of matrices.

Another observation is that even though the joint and generalized spectral radius can be defined also for unbounded families the first equality does not hold in general. Consider for example the unbounded family:

$$\mathscr{F} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \ldots \right\}$$

For this family since every product of the two matrices is upper triangular with ones in the main diagonal it is evident that $\overline{\rho}(\mathscr{F}) = 1$ and obviously $\widehat{\rho}(\mathscr{F}) = +\infty$ since the family is unbounded (see [114] for details and [70] for another example).

We observe also that Gurvits in [84] give a counterexample to the first equality in the case of two operators in an infinite dimensional Hilbert space.

From now on and if not differently specified we will always consider bounded sets of matrices. Theorem 4.2.7 implies that we can simply refer to the *spectral radius* $\rho(\mathscr{F})$ of the family of matrices $\mathscr{F}$.

4.2.8 Definition (Trajectory)**.** *Given a family $\mathscr{F} = \{A_i\}_{i\in\mathscr{I}}$, we define, for an arbitrary nonzero vector $x \in \mathbb{C}^n$, the* trajectory

$$\mathscr{T}[\mathscr{F},x] = \{Px \ \mid P \in \mathscr{P}(\mathscr{F})\} \tag{4.30}$$

as the set of vectors obtained by applying all the products P in the multiplicative semigroup $\mathscr{P}(\mathscr{F})$ to the vector x.

4.2.9 Definition (Discrete linear inclusion)**.** *The* discrete linear inclusion *is the set of all the trajectories associated with all the possible vectors in $\mathbb{C}^n$. This set is denoted by DLI($\mathscr{F}$).*

Properties

We resume now properties valid for the spectral radius of a bounded set of matrices $\mathscr{F} = \{A_i\}_{i\in\mathscr{I}}$

1. **Multiplication by a scalar:** For any set $\mathscr{F}$ and for any number $\alpha \in \mathbb{C}$

$$\rho(\alpha\mathscr{F}) = |\alpha|\rho(\mathscr{F}) \tag{4.31}$$

2. **Continuity:** The joint spectral radius is continuous in its entries as established by Heil and Strang [87]. Wirth has proved [118] that the joint spectral radius is even locally Lipschitz continuous on the space of compact irreducible sets of matrices (an explicit formula for the related Lipschitz constant has been evaluated by Kozyakin [95]).

3. **Powers of the family:** For any set $\mathscr{F}$ and for any $k \geq 1$

$$\rho(\mathscr{F}^k) \leq \rho^k(\mathscr{F})$$

4. **Invariance under similarity:** The spectral radius of the family is invariant under similarity transformation, so for any set of matrices $\mathscr{F}$, and any invertible matrix T

$$\rho(\mathscr{F}) = \rho(T\mathscr{F}T^{-1}) \tag{4.32}$$

This because to any product $A_1 \cdot \ldots \cdot A_k \in \mathscr{P}_k(\mathscr{F})$ corresponds a product $T \cdot A_1 \cdot \ldots \cdot A_k \cdot T^{-1} \in \mathscr{P}_k(T\mathscr{F}T^{-1})$ with equal spectral radius.

5. **Conjugate or transposed family:** The conjugate or transposed family (family obtained taking the conjugate/transpose of every matrix in the original set) has the same spectral radius as the original one [76, Lemma 5.1]

$$\rho(\mathscr{F}^*) = \rho(\mathscr{F}) \qquad \rho(\mathscr{F}^T) = \rho(\mathscr{F}) \tag{4.33}$$

6. **Block triangular matrices:** Given a family of block upper triangular matrices

$$\mathscr{F} = \left\{ \begin{pmatrix} A_i & B_i \\ 0 & C_i \end{pmatrix} \right\}_{i \in \mathscr{I}}$$

we have that

$$\rho(\mathscr{F}) = \max\left\{\rho(\{A_i\}_{i\in\mathscr{I}}), \rho(\{C_i\}_{i\in\mathscr{I}})\right\}. \tag{4.34}$$

This is due to the closure, with respect to the multiplication, of block upper triangularity [52, Lemma II (c)]. Clearly the same holds for lower triangular matrices. This result generalizes to the case of more than two blocks on the diagonal.

7. **Closure and convex hull:** The closure and the convex hull of a set have the same spectral radius of the original set

$$\rho(\mathrm{conv}\mathscr{F}) = \rho(\mathrm{cl}\,\mathscr{F}) = \rho(\mathscr{F}) \tag{4.35}$$

This result was first obtained by Barabanov in 1988 [51]. An alternative proof, given by Theys in [114, page 17], is based on the common spectral radius definition (4.28) and the property

$$\sup_{A_i \in \mathscr{F}} \|A_i\| = \sup_{A_i \in \mathrm{conv}\mathscr{F}} \|A_i\| = \sup_{A_i \in \mathrm{cl}\,\mathscr{F}} \|A_i\|. \tag{4.36}$$

8. **Uniform asymptotic stability characterization [52, Theorem I (b)]:** For any bounded set of matrices $\mathscr{F}$ and for any $k \geq 1$, all matrix products $P \in \mathscr{P}_k(\mathscr{F})$ converge to the zero matrix as $k \to \infty$, i.e.

$\mathscr{F}$ is uniformly asymptotically stable (ref page 108), if and only if $\rho(\mathscr{F}) < 1$.

In other words the spectral radius of the family of matrices $\mathscr{F}$ gives information about the uniform asymptotic stability of the associated dynamical system DLI($\mathscr{F}$), defined on page 113.

9. Product boundedness [52, Theorem I (a)]: Given a bounded set of matrices $\mathscr{F}$, if products $P \in \mathscr{P}_k(\mathscr{F})$, $k \in \mathbb{N}$, converge as $k \to \infty$. Then, the multiplicative monoid $\mathscr{P}^*(\mathscr{F})$ defined in (4.20) is bounded and $\rho(\mathscr{F}) \leq 1$.

The opposite implication is not true in general:

Given a defective family with $\rho(\mathscr{F}) = 1$, products $P \in \mathscr{P}_k(\mathscr{F})$, $k \in \mathbb{N}$, explode for $k \to \infty$ by Definition 4.2.11.

We return on this aspect in the following Chapter on page 138.

10. Special cases:

1. Recalling that the set of stochastic matrices is closed under matrix multiplication and that every stochastic matrix has spectral radius equal 1 (ref Section 4.1), if the matrices in $\mathscr{F}$ are all stochastic then the spectral radius of the family is exactly 1.

2. If the matrices in $\mathscr{F}$ are all upper–triangular, if they can be simultaneously upper–triangularized, if all the matrices in $\mathscr{F}$ commutes or, more in general, if the Lie algebra associated with the set of matrices is solvable (commutative families are *Abelian Lie algebras* which are always solvable), if they are all symmetric or, more in general, if they are all normal or, finally, if they can be simultaneously normalized, then

$$\rho(\mathscr{F}) = \max_{A_i \in \mathscr{F}} \{\rho(A_i)\} \tag{4.37}$$

   For more details see [84, 74, 69, 114, 90].

3. If $\mathscr{F} = \{A, A^*\}$ then $\rho(\mathscr{F}) = \rho(AA^*)^{1/2} = \sigma_1(A)$ i.e. the largest singular value of A. In fact [114, Proposition 6.20] using the four

members inequality (4.23) for $k=2$ we have

$$\rho(AA^*)^{1/2} = \sigma_1(A) = \|AA^*\|_2^{1/2} \tag{4.38}$$

4. ([104] and [75, Theorem 4]). Consider the family $\mathscr{F} = \{A, B\}$ with

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B := \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \quad a,b,c,d \in \mathbb{R}$$

The joint spectral radius of the family $\mathscr{F}$ is given by

$$\rho(\mathscr{F}) = \begin{cases} \rho(A) = \rho(B) & \text{if} \quad bc \geq 0 \\ \sqrt{\rho(AB)} & \text{if} \quad bc < 0 \end{cases}$$

5. ([104] and [75, Theorem 5]). Consider the family $\mathscr{F} = \{A, B\}$ with

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B := \begin{pmatrix} d & c \\ b & a \end{pmatrix} \quad a,b,c,d \in \mathbb{R}$$

The joint spectral radius of the family $\mathscr{F}$ is given by

$$\rho(\mathscr{F}) = \begin{cases} \rho(A) = \rho(B) & \text{if} \quad |a-d| \geq |b-c| \\ \sqrt{\rho(AB)} & \text{if} \quad |a-d| < |b-c| \end{cases}$$

6. Let $|\mathscr{F}|$ be the family of matrices obtained from $\mathscr{F}$ as follows:

$$A = [a_{ij}] \in \mathscr{F} \quad \longrightarrow \quad |A| = [|a_{ij}|] \in |\mathscr{F}|.$$

Then

$$\rho(|\mathscr{F}|) \geq \rho(\mathscr{F}) \tag{4.39}$$

From the previous result and the four members inequality (4.23) we have that

$$\overline{\rho}_k(\mathscr{F}) \leq \rho(\mathscr{F}) \leq \rho(|\mathscr{F}|)$$

So if $P \in \mathscr{P}_k(\mathscr{F})$, $k \in \mathbb{N}$, is such that $\rho(P)^{1/k} = \rho\left(|\mathscr{F}|\right)$, then

$$\rho\left(\mathscr{F}\right) = \rho(P)^{1/k}.$$

11. Non–algebraicity: Any set composed of k real $n\times n$–matrices can be seen as a point in the space $\mathbb{R}^{kn^2}$. Therefore, a subset of $\mathbb{R}^{kn^2}$ is a set of k–tuples of $n\times n$–matrices. Given a subset of $\mathbb{R}^{kn^2}$ this is defined *semi–algebraic* if it is a finite union of sets that can be expressed by a finite list of polynomial equalities and inequalities. Kozyakin [93] has shown that, for all $k,n\geq 2$, the set of points $x\in\mathbb{R}^{kn^2}$ such that $\rho(x)<1$ is not semi–algebraic and, for all $k,n\geq 2$, the set of points $x\in\mathbb{R}^{kn^2}$ corresponding to a bounded semigroup $\mathscr{P}(x)$ is not semi–algebraic (the original paper by Kozyakin contains a flaw and the correction has been published by the same author only in Russian. For a corrected version in English we refer the reader to the Doctoral work of Theys [114, Section 4.2]). In practice in the general case, given a discrete linear inclusion DLI($\mathscr{F}$), there is no procedure involving a finite number of operations that allows to decide whether DLI($\mathscr{F}$) is uniformly asymptotically stable or not i.e. the uniform asymptotic stability of DLI($\mathscr{F}$) is in general hard to determine.

12. NP–hardness: In [116] Tsitsiklis and Blondel proved that, given a set of two matrices $\mathscr{F}$ and unless P = NP, the spectral radius $\rho(\mathscr{F})$ is not polynomial–time approximable. This holds true even if all nonzero entries of the two matrices are constrained to be equal. Let us recall that the function $\rho(\mathscr{F})$ is *polynomial–time approximable* if there exists an algorithm $\rho^*(\mathscr{F},\varepsilon)$, which, for every rational number $\varepsilon>0$ and every set of matrices $\mathscr{F}$ with $\rho(\mathscr{F})>0$, returns an approximation of $\rho(\mathscr{F})$ with a relative error of at most ε (i.e. such that $|\rho^*-\rho|\leq\varepsilon\rho$) in time polynomial in the bit size of $\mathscr{F}$ and ε (if $\varepsilon=p/q$, with p and q relatively prime numbers, its *bit size* is equal to $\log(pq)$); however there are algorithms which are polynomial either in the bit size of $\mathscr{F}$ or in ε. We conclude that the computation of the spectral radius of a set of matrices is in general *NP–hard* and, consequently, it is NP–hard to decide the stability of all products of a set of matrices (for a survey of NP–hardness and undecidability we refer the reader to [62]). We observe here that Gurvits in [85] provides a polynomial–time algorithm for the case of binary matrices.

13. Undecidability: A decision problem is a problem which output is binary and can be interpreted as “yes” or “not”. For instance the problem of deciding whether an integer matrix is nonsingular is a decision problem. Since the nonsingularity can be checked, for example, by computing the determinant of the matrix and comparing it to zero it is a *decidable problem*, i.e. a problem for which there exists an algorithm that always halts with the right answer. But there are also problems for which this kind of algorithm does not exist, these are *undecidable problems.*

Given a set of matrices $\mathscr{F}$:

- The problem of determining if the semigroup $\mathscr{P}(\mathscr{F})$ is bounded is undecidable
- The problem of determining if $\rho(\mathscr{F}) \leq 1$ is undecidable

These two results, which remain true even if $\mathscr{F}$ contains only rational entries [63, 54], teach us that does not exist any algorithm allowing to compute the spectral radius of a generic set $\mathscr{F}$ in finite time.

It is still unknown if it does exist in the generic case an algorithm that, given a finite set of matrices $\mathscr{F}$, decides whether $\rho(\mathscr{F}) < 1$. Such an algorithm would allow to check the uniform asymptotic stability of the dynamical system ruled by the generic set $\mathscr{F}$. In the following we discuss the relation between this kind of algorithm and the so–called finiteness property.

The actual computation of $\rho(\mathscr{F})$ is an important problem in several applications, as we mentioned in the introduction of the present Chapter. According to the previous properties of non–algebraicity, NP–hardness and undecidability the problem appears quite difficult in general.

However, this is not reason enough for declaring the problem intractable and refraining from further research. As we discover in the next subsection the existence of an s.m.p. for the family (i.e. a product in the semigroup $\mathscr{P}(\mathscr{F})$ with particular properties) allows in the general case to evaluate exactly the spectral radius of a family making use of the Definition 4.2.6 as an actual computational tool. In order to do this we need the inf in equation (4.28) to be a min, but this is always true for irreducible families.

4.2.3 Irreducibility, nondefectivity and finiteness property

When the inf in (4.28) is a min we say that the family $\mathscr{F} = \{A_i\}_{i\in\mathscr{I}}$ admits an *extremal norm*.

4.2.10 Definition (Extremal norm)**.** *A norm $\|\cdot\|_*$ satisfying the condition*

$$\rho(\mathscr{F}) = \|\mathscr{F}\|_* := \sup_{i\in\mathscr{I}} \|A_i\|_*$$

is said to be extremal *for the family $\mathscr{F}$ (for an extended discussion see [119]).*

Equivalently a norm $\|\cdot\|_*$ is called extremal for a given set $\mathscr{F} = \{A_i\}_{i\in\mathscr{I}}$ if it satisfies $\|A_i\|_* \le \rho(\mathscr{F})$ for every $i \in \mathscr{I}$.

From Proposition 4.2.4 it is clear that, for a given norm, this inequality cannot be strict simultaneously for all the matrices in the set.

Given a bounded family $\mathscr{F} = \{A_i\}_{i\in\mathscr{I}}$ of $n\times n$–matrices with $\rho(\mathscr{F}) > 0$, the *normalized family* is given by

$$\mathscr{F}^* = \{A_i/\rho(\mathscr{F})\}_{i\in\mathscr{I}} \tag{4.40}$$

with spectral radius $\rho(\mathscr{F}^*) = 1$ and $\mathscr{P}(\mathscr{F}^*)$ is the associated multiplicative semigroup (ref equation (4.19) on page 109).

The definition of (weakly) defective matrix, given in section 4.1, extends to bounded families of matrices as follows:

4.2.11 Definition (Defective and Nondefective Families)**.** *A bounded family $\mathscr{F}$ of $n\times n$–matrices is said to be* defective *if the corresponding normalized family $\mathscr{F}^*$ is such that the associated semigroup $\mathscr{P}(\mathscr{F}^*)$ is an unbounded set of matrices. Otherwise, if either $\rho(\mathscr{F}) = 0$ or $\rho(\mathscr{F}) > 0$ with $\mathscr{P}(\mathscr{F}^*)$ bounded, then the family $\mathscr{F}$ is said to be* nondefective.

The following result can be found, for example, in [110] and [52]:

4.2.12 Proposition. *A bounded family $\mathscr{F}$ of $n\times n$–matrices admits an extremal norm $\|\cdot\|_*$ if and only if it is nondefective.*

As previously mentioned we want to make use of Definition 4.2.6 as an actual computational tool for the spectral radius $\rho(\mathscr{F})$. To do this we need to ensure that the family admits an *extremal norm* i.e. we have to check the defectivity or nondefectivity of the set $\mathscr{F}$.

Strictly connected to defectivity of a family there is the concept of reducibility.

4.2.13 Definition (Reducible and Irreducible families). *A bounded family $\mathscr{F} = \{A_i\}_{i\in\mathscr{I}}$ of $n\times n$–matrices is said to be* reducible *if there exist a nonsingular $n\times n$–matrix M and two integers $n_1, n_2 \geq 1$, $n_1 + n_2 = n$, such that*

$$M^{-1}A_iM = \begin{pmatrix} A_i^{(11)} & A_i^{(12)} \\ O & A_i^{(22)} \end{pmatrix} \quad \textit{for all} \quad i \in \mathscr{I} \tag{4.41}$$

where the blocks $A_i^{(11)}$, $A_i^{(12)}$, $A_i^{(22)}$ are $n_1\times n_1$–, $n_1\times n_2$– and $n_2\times n_2$–matrices, respectively. On the contrary, if a family $\mathscr{F}$ is not reducible, then it is said to be irreducible.

Irreducibility means that only the trivial subspaces 0 and $\mathbb{C}^n$ are invariant under all the matrices of the family $\mathscr{F}$. Otherwise $\mathscr{F}$ is called reducible. The concept of irreducibility was introduced in the joint spectral radius theory by Barabanov in [51], where he named irreducible families *nonsingular sets*.

We observe that some authors refer to reducibility as decomposability (irreducibility as non–decomposability) in order to avoid confusion with the notion of reducibility commonly used in linear algebra and utilized in the previous Chapters [24, Definition 6.2.21].

An immediate consequence of irreducibility of $\mathscr{F}$ is that $\rho(\mathscr{F}) > 0$, in fact, in this case the semigroup $\mathscr{P}(\mathscr{F})$ is irreducible and, therefore, does not consist of nilpotent elements, by the Levitzky Theorem [97]. So we can always normalize an irreducible set of matrices $\mathscr{F}$ by $\rho(\mathscr{F})$ obtaining a set with generalized spectral radius equal to 1.

Another consequence of irreducibility of a family is stated in the next Theorem and its Corollary, which follow easily from the Barabanov's construction of extremal norms for irreducible families of matrices [51].

4.2.14 Theorem ([72, Lemma 4]). *If a bounded family $\mathscr{F}$ of $n \times n$–matrices is defective, then is reducible.*

and, therefore,

4.2.15 Corollary ([51, 72, 107]). *If a bounded family of matrices is irreducible then it is nondefective, i.e. it does exist an extremal norm for the family.*

In Figure 4.1 it is represented the space $\mathscr{B}$ of bounded families of matrices in $\mathbb{C}^{n \times n}$. This space can be split into the set of the reducible families $\mathscr{R}$ and its complement $\mathscr{I}_{\mathscr{R}}$, the set of the irreducible ones. Families of matrices can be nondefective or defective: the set $\mathscr{D}$ of the defective families is a proper subset of $\mathscr{R}$ i.e. $\mathscr{D} \subsetneq \mathscr{R}$. In fact Theorem 4.2.14 implies that a defective family is always reducible, but the opposite implication is not necessarily true. For example, for $n \geq 2$ all single families $\mathscr{F} = \{A\}$ are clearly reducible as the Jordan canonical form proves, but not necessarily defective. The set of nondefective families $\mathscr{N}_{\mathscr{D}}$, the complement of $\mathscr{D}$ in $\mathscr{B}$, is denoted by grey dots.

About the dimension of set $\mathscr{D}$ and $\mathscr{R}$ Maesumi [100] proposed the following conjecture

4.2.16 Conjecture. *Reducible (decomposable) matrix sets form a set of measure zero in the corresponding space of matrices. Defective matrix sets form a set of measure zero within the set of reducible matrices.*

In the next Chapter, on page 137, we delve further this analysis especially explaining how reducible families can be handled.

We add just that Brayton and Tong in [65] give an alternative sufficient–condition for nondefectiveness. They prove that, considered each matrix P in the semigroup $\mathscr{P}(\mathscr{F})$ and the associated similarity matrix S_P that reduce P into its Jordan form, if every S_P has columns linearly independent uniformly on all $P \in \mathscr{P}(\mathscr{F})$, then $\mathscr{F}$ is nondefective. This alternative sufficient–condition represents the generalization of the concept of strongly nondefectiveness to the case of sets of matrices, in fact for a single matrix A strongly nondefectiveness is equivalent to semisimplicity of all the eigenvalues in the spectrum of A or equivalently to diagonalizability of A (ref pages

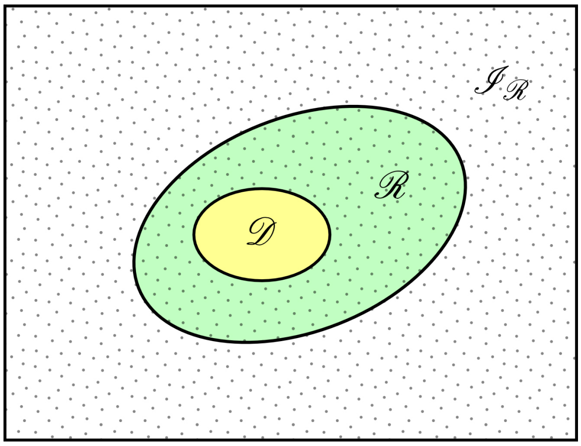


Figure 4.1: Space of bounded families of matrices $\mathscr{B}$: the set of defective families is denoted by $\mathscr{D}$, while its complement, highlighted by dots, is the set of nondefective families $\mathscr{N}_{\mathscr{D}}$.

5 and 105). Clearly strongly nondefectiveness implies nondefectiveness, but checking this sufficient–condition is not feasible in practice.

As previously mentioned there are not known algorithms for deciding uniform asymptotic stability of a generic set of matrices and it is unknown if this problem is algorithmically decidable in general. We have also seen that uniform asymptotic stability of the set $\mathscr{F}$ is equivalent to $\rho(\mathscr{F}) < 1$. In order to check if $\rho(\mathscr{F}) < 1$ for finite families we may think of using the four members inequality (4.23)

$$\overline{\rho}_k(\mathscr{F}) \leq \rho(\mathscr{F}) \leq \widehat{\rho}_k(\mathscr{F}) \qquad \text{for all } k \geq 1$$

The procedure could be the following [70]:

(1) We evaluate

$$\overline{\rho}_k(\mathscr{F}) := \max_{P \in \mathscr{P}_k(\mathscr{F})} \rho(P)^{1/k} \quad \text{and} \quad \widehat{\rho}_k(\mathscr{F}) := \max_{P \in \mathscr{P}_k(\mathscr{F})} \|P\|^{1/k}$$

for increasing values of $k \geq 1$.

(2) As soon as $\widehat{\rho}_k < 1$ or $\overline{\rho}_k \geq 1$ we stop and declare the set uniform asymptotic stable or unstable respectively.

We observe that this procedure always stops after finitely many steps unless $\rho = 1$ and $\overline{\rho}_k < 1$ for all $k \geq 1$, but this never occurs for families that, satisfying the finiteness property, have an s.m.p.

4.2.17 Definition (Finiteness property and s.m.p.)**.** *A finite family $\mathscr{F}$ of $n \times n$–matrices has the* finiteness property *if there exists, for some $k \geq 1$, a product $\overline{P} \in \mathscr{P}_k(\mathscr{F})$ such that*

$$\rho\left(\overline{P}\right) = \rho\left(\mathscr{F}\right)^k.$$

The product $\overline{P}$ is said to be a spectrum–maximizing product *or* s.m.p. *for $\mathscr{F}$. Some authors refer to* optimal product *instead of s.m.p., see for instance [91, 100].*

An s.m.p. is said minimal *if it is not a power of another s.m.p. of $\mathscr{F}$.*

Any eigenvector $x \neq 0$ of $\overline{P}$ related to an eigenvalue λ with $|\lambda| = \rho(\overline{P})$ is said to be a leading eigenvector *of $\mathscr{F}$.*

From the previous definition is evident that uniform asymptotic stability is algorithmically decidable for finite sets of matrices that have the finiteness property.

Lagarias and Wang in 1995 [96] conjectured that the finiteness property was valid for all finite families of real matrices (the so–called *finiteness conjecture*). Unfortunately this conjecture does not hold true: Bousch and Mairesse [64] and later other authors [60, 94] presented non–constructive counterproofs. In particular in [60] Blondel et al. proved that for the parametric family

$$\mathscr{F}_\alpha = \{A,\ \alpha B\} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \alpha \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\} \quad \text{with} \quad \alpha \in [0,1]$$

there exist uncountably many values of the parameter α for which $\mathscr{F}_\alpha$ does not satisfy the finiteness conjecture. They were unable to find a single explicit value of α and they conjectured that the set of values $\alpha \in [0,1]$ for which the finiteness conjecture is not satisfied is of measure zero. Recently Hare et al. [86], using combinatorial ideas and ergodic theory, have been able to approximate, up to a desired precision, an explicit value α such that $\mathscr{F}_\alpha$ does not satisfy the finiteness conjecture. The question if there exist families of matrices with rational entries that violate the conjecture remains still open. Based on all the numerical experiments developed in the last years and the results previously mentioned a new conjecture has been introduced:

4.2.18 Conjecture ([60, 100, Blondel et al. and Maesumi])**.** *The finiteness property is true a.e. in the space of finite families of complex square matrices, i.e. the set of families of matrices for which the finiteness property is not true has measure zero in the space of finite families.*

If this conjecture is true then it suggests us to track s.m.p.'s candidates out and validate them with some procedure in order to find the spectral radius of the family. In the next Chapter we explain how to perform the validation step using particular extremal norms for the given set.

The idea behind this last conjecture is that the NP–hardness, non–algebraicity and undecidability results are due to certain rare and extreme cases and that in the generic case the evaluation of the spectral radius, while could be computationally intensive, is possible. About the computational complexity we remind an example, given by Berger and Wang [52, Example 2.1], of a set of two 2×2–matrices with minimal s.m.p. of length $k \geq 1$ with k arbitrarily large:

$$\mathscr{F} = \left\{ \alpha^k \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \alpha^{-1} \begin{pmatrix} \cos\frac{\pi}{2k} & \sin\frac{\pi}{2k} \\ -\sin\frac{\pi}{2k} & \cos\frac{\pi}{2k} \end{pmatrix} \right\} \quad \text{with} \quad 1 < \alpha < \left(\cos\frac{\pi}{2k}\right)^{-1}$$

They prove that $\rho(\mathscr{F}) = 1$, $\overline{\rho}_j(\mathscr{F}) < 1$ for $j \leq k$ and $\overline{\rho}_{k+1}(\mathscr{F}) = 1$.

We recall that Blondel and Tsitsiklis in [63] proved also that the *effective finiteness conjecture* is false:

4.2.19 Conjecture (Effective finiteness conjecture)**.** *For any finite set $\mathscr{F}$ of square matrices with rational entries there exists an effectively computable natural number $t(\mathscr{F})$ such that $\overline{\rho}_{t(\mathscr{F})}(\mathscr{F}) = \rho(\mathscr{F})$*

The falseness of this conjecture implies that, given a family of matrices with rational entries which admits a spectrum–maximizing product, the length of the s.m.p. can be arbitrary long and consequently the computation of the spectral radius can become a tough problem. Nevertheless for random families this product appears to be, luckily, quite short in general.

The finiteness property is known to hold in many cases:

- when the matrices in $\mathscr{F}$ are all simultaneously upper–triangularizable, or they can be simultaneously normalized, or the Lie algebra associated with the set $\mathscr{F}$ is solvable. In these cases, in fact, the spectral radius is simply equal to the maximum of the spectral radii of the matrices (Property 10, special case 2 on page 115);

- when a finite set of real matrices admits an extremal *piecewise analytic norm* in $\mathbb{R}^n$. A piecewise analytic norm is any norm on $\mathbb{R}^n$ whose unit ball $\boldsymbol{B}$ has a boundary which is contained in the zero set of a holomorphic function $f(z)$, i.e. complex differentiable at every point in its domain, defined on a connected open set $\Omega \in \mathbb{C}^n$ containing 0, which has $f(0) \neq 0$ (Lagarias and Wang [96]);

- when a finite set of real matrices admits an extremal *piecewise algebraic norm* in $\mathbb{R}^n$. A piecewise algebraic norm is one whose boundary is contained in the zero set of a polynomial $p(z) \in \mathbb{R}[z_l, \dots, z_n]$, which has $p(0) \neq 0$. This is the case when the unit ball of a norm is a polytope (see the next Chapter), or an ellipsoid (ref page 104), or the l^p norm for rational p, with $1 \leq p \leq \infty$ (Lagarias and Wang in [96] extended the result proved by Gurvits in [84] for real polytope extremal norms to the general case of piecewise algebraic norms in $\mathbb{R}^n$);

- when a finite set of matrices admits a complex polytope extremal norm. This it has been proved by Guglielmi, Wirth and Zennaro in [76, Theorem 5.1] extending to the complex case the results by Gurvits [84] and Lagarias and Wang [96]. We come back to polytope norms in the next Chapter.

For other classes of sets of matrices the finiteness property has been only conjectured to be true, an example is the class of sets of matrices with rational entries. Indeed the proof of the finiteness property for sets of rational matrices would be satisfactory for practical applications: the matrices that one handles or enters in a computer are rational–valued.

Recently Blondel and Jungers [91] have proved the following Theorem:

4.2.20 Theorem ([91, Theorem 4])**.**

1. *The finiteness property holds for all sets of nonnegative rational matrices if and only if it holds for all pairs of binary matrices.*
2. *The finiteness property holds for all sets of rational matrices if and only if it holds for all pairs of matrices with entries in* $\{-1, 0, +1\}$.

They proposed, consequently, the following conjecture

4.2.21 Conjecture ([56, 91, Blondel, Jungers and Protasov])**.** *Pairs of binary matrices have the finiteness property.*

If this conjecture is correct then, by Theorem 4.2.20, nonnegative rational matrices also satisfy the finiteness property and, thus, the question $\rho(\mathscr{F}) < 1$ becomes decidable for sets of matrices with nonnegative rational entries. From a decidability point of view this last result would be somewhat surprising since it is known that the closely related question $\rho(\mathscr{F}) \leq 1$ is known to be no algorithmically decidable for such sets of matrices (ref Property 13 on page 117). Blondel and Jungers [91] proved that pairs of 2×2 binary–matrices satisfy the finiteness property and observed that the length of the s.m.p.'s is always very short. This result is promising even though a generalization to the case of $n\times n$–matrices seems quite difficult due to the falseness of the effective finiteness conjecture 4.2.19, which implies that the length of the s.m.p.'s for families of $n\times n$–matrices can become extremely long.

A more general version of the previous Conjecture is the following

4.2.22 Conjecture ([56, 91, Blondel, Jungers and Protasov])**.** *The finiteness property holds for pairs of matrices with entries in* $\{-1, 0, +1\}$ *(the so–called sign–matrices).*

This last would imply, by Theorem 4.2.20, that the finiteness property holds for all sets of rational matrices. In the following Chapter we prove analytically the finiteness property for pairs of 2×2 sign–matrices, i.e. matrices with entries in $\{-1, 0, +1\}$.

Chapter 5

An algorithm for the Spectral Radius exact computation

An algorithm for efficiently computing lower and upper bounds for the spectral radius $\rho(\mathscr{F})$ was proposed by Gripenberg in [74] and is based on the four members inequality (4.23) applied to bounded families

$$\overline{\rho}_k(\mathscr{F}) \le \rho(\mathscr{F}) \le \widehat{\rho}_k(\mathscr{F}) \quad \text{for all } k \ge 1 \tag{5.1}$$

plus a branch and bound technique. Some other approaches for the approximation of this quantity have been recently considered for example in [108, 58, 59] and [57]. For an overview of several recent algorithms and methods that use the extremal norm approach for the approximation of the joint spectral radius we refer the reader to the recent paper [55] by Chang and Blondel and to the theses of Theys [114] and Jungers [90]. The problem we handle here, however, is that of an exact computation.

A way to compute exactly the joint spectral radius is based on the following property. If $\alpha > 0$ then by (4.31) we have that

$$\rho(\mathscr{F}) = \alpha\rho\left(\frac{1}{\alpha}\mathscr{F}\right).$$

So, if $Q \in \mathscr{P}_k(\mathscr{F})$, $k \ge 1$, is a certain product such that $\alpha = \rho(Q)^{1/k} > 0$, then we have $\rho\left(\frac{1}{\alpha}\mathscr{F}\right) \ge 1$. If we are able to find a norm such that $\|\frac{1}{\alpha}\mathscr{F}\| = 1$, then

$$\alpha \leq \rho(\mathscr{F}) \leq \alpha \quad \Longrightarrow \quad \rho(\mathscr{F}) = \alpha = \rho(Q)^{1/k}.$$

This would mean that the finiteness property holds and the product Q is an s.m.p. for the family $\mathscr{F}$. The key point is the individuation of an extremal norm.

In this Chapter we present an algorithm, based on the previous idea, for the exact computation of the spectral radius for finite sets of matrices via the construction of the unit ball of an extremal *polytope norm*.

As an application we address the problem of establishing the finiteness property of pairs of 2×2 sign–matrices. This problem is related to the conjecture that pairs of sign–matrices fulfil the finiteness property for any dimension (ref conjecture 4.2.22). The truthfulness of this conjecture would imply, as we mentioned in the previous Chapter, that finite sets of rational matrices fulfil the finiteness property, which would be very important in terms of the computation of the *joint spectral radius*. The technique used in this Chapter could suggest an extension of the analysis to $n\times n$ sign–matrices, which still remains an open problem.

The summary of the Chapter is the following. In Section 5.1, after recalling some definitions and results on polytope norms, we introduce the main ideas of a procedure able to find an extremal norm in this class. We provide a set of assumptions which guarantees the existence of an extremal norm of polytope type that is finitely generated(see also [76]) and we present the algorithm. Then, in Section 5.2, we prove the finiteness property for pairs of 2×2 sign–matrices on a case–by–case basis. Last Section is devoted to outline some conclusions, while Appendix A contains the details of the analysis presented in Section 5.2.

5.1 Extremal polytope norms

In this section we focus our attention on a special class of norms, in particular we are concerned with the possible construction of the unit ball of an extremal norm for a finite family.

We start recalling basic definitions and properties for real polytope norms (see for instance [120]).

5.1.1 Definition (Balanced real polytope (b.r.p.))**.** *A bounded set* $\mathscr{P} \subset \mathbb{R}^n$ *is a* balanced real polytope (b.r.p.) *if there exists a finite set of vectors* $\mathscr{X} = \{x_i\}_{i=1}^m$ *(with* $m \geq n$*) such that* $\mathrm{span}(\mathscr{X}) = \mathbb{R}^n$*, i.e. the set* $\mathscr{X}$ *is absorbing, and*

$$\mathscr{P} = \mathrm{co}(\mathscr{X}, -\mathscr{X}), \tag{5.2}$$

where co *denotes the convex hull. Therefore*

$$\mathscr{P} = \left\{ x = \sum_{i=1}^m \lambda_i x_i + \mu_i(-x_i) \ : \ \lambda_i, \mu_i \geq 0 \text{ and } \sum_{i=1}^m (\lambda_i + \mu_i) \leq 1 \right\} \tag{5.3}$$

Moreover, if $\mathrm{co}(\mathscr{X}', -\mathscr{X}') \subsetneq \mathrm{co}(\mathscr{X}, -\mathscr{X})\ \forall\ \mathscr{X}' \subsetneq \mathscr{X}$, then the set $\mathscr{X}$ is called an *essential system of vertices* for $\mathscr{P}$ and any vector $x_i \in \mathscr{X}$ is called a *vertex* of $\mathscr{P}$.

Since the set $\mathscr{P}$ is absorbing, convex and bounded, clearly, it is the unit ball of a norm $\|\cdot\|_{\mathscr{P}}$ on $\mathbb{R}^n$.

5.1.2 Definition (Real polytope norm)**.** *We call* real polytope norm *any norm* $\|\cdot\|_{\mathscr{P}}$ *whose unit ball is a b.r.p.* $\mathscr{P}$*.*

The real polytope norms are characterized as follows.

5.1.3 Lemma. *Let* $\mathscr{X} = \{x_i\}_{i=1}^m$ *be a set of vectors spanning* $\mathbb{R}^n$ *and* $\mathscr{P} = \mathrm{co}(\mathscr{X}, -\mathscr{X})$*. Set* $\|\cdot\|_{\mathscr{P}}$ *the corresponding real polytope norm. Then,* $\forall z \in \mathbb{R}^n$*, we have*

$$\|z\|_{\mathscr{P}} = \min_{\lambda_i \geq 0, \mu_i \geq 0} \left\{ \sum_{i=1}^m (\lambda_i + \mu_i) \ : \ z = \sum_{i=1}^m \lambda_i x_i + \mu_i(-x_i) \right\}. \tag{5.4}$$

Proof. The equality (5.4) is just the *Minkowski functional* (see [88]) associated with the set $\mathscr{P}$ defined in (5.3). □

Note that (5.4) is a linear programming problem, which can be solved efficiently (see e.g. [117]).

We consider now the complex case. Following [82] we define

5.1.4 Definition (Absolutely convex set)**.** *Given a set* $\mathscr{X} \subset \mathbb{C}^n$*, it is defined* absolutely convex *if, for all* $x_1, x_2 \in \mathscr{X}$ *and* $\lambda_1, \lambda_2 \in \mathbb{C}$ *s.t.* $|\lambda_1| + |\lambda_2| \leq 1$*, it holds that* $\lambda_1 x_1 + \lambda_2 x_2 \in \mathscr{X}$*.*

5.1.5 Definition (Absolute convex hull (absco))**.** *Let* $\mathscr{X} \subset \mathbb{C}^n$*. Then the intersection of all absolutely convex sets containing* $\mathscr{X}$ *it is called the* absolutely convex hull *of* $\mathscr{X}$ *and denoted by* $\mathsf{absco}(\mathscr{X})$*.*

It is well known that $\mathsf{absco}(\mathscr{X})$ is the set of all the finite absolutely convex linear combinations of vectors of $\mathscr{X}$, i.e. $x \in \mathsf{absco}(\mathscr{X})$ if and only if there exist $x_1, \dots, x_k \in \mathscr{X}$ with $k \geq 1$ such that

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with} \quad \lambda_i' \text{s} \in \mathbb{C} \quad \text{and} \quad \sum_{i=1}^{k} |\lambda_i| \leq 1. \tag{5.5}$$

So if $\mathscr{X} = \{x_1, \dots, x_m\}_{m \in \mathbb{N}}$ is a finite set of vectors in $\mathbb{C}^n$, then

$$\mathsf{absco}(\mathscr{X}) = \left\{ x \in \mathbb{C}^n \;\middle|\; x = \sum_{i=1}^{m} \lambda_i x_i \text{ with } \lambda_i' \text{s} \in \mathbb{C} \text{ and } \sum_{i=1}^{m} |\lambda_i| \leq 1 \right\} \tag{5.6}$$

and, in this case, it is a closed subset of $\mathbb{C}^n$.

We can now generalize the usual concept of real balanced polytope to the complex case.

5.1.6 Definition (Balanced complex polytope (b.c.p.))**.** *We say that a bounded set* $\mathscr{P} \subset \mathbb{C}^n$ *is a* balanced complex polytope (b.c.p.) *if there exists a finite set of vectors* $\mathscr{X} = \{x_i\}_{i=1}^m$ *such that* $\mathsf{span}(\mathscr{X}) = \mathbb{C}^n$ *and*

$$\mathscr{P} = \mathsf{absco}(\mathscr{X}). \tag{5.7}$$

Moreover, if $\mathsf{absco}(\mathscr{X}') \subsetneq \mathsf{absco}(\mathscr{X})$ *for all* $\mathscr{X}' \subsetneq \mathscr{X}$*, then* $\mathscr{X}$ *will be called an* essential system of vertices *for* $\mathscr{P}$*, whereas any vector* ux_i *with* $u \in \mathbb{C}$*,* $x_i \in \mathscr{X}$*, and* $|u| = 1$*, will be called a* vertex *of* $\mathscr{P}$*.*

We observe that, geometrically speaking, a b.c.p. $\mathscr{P}$ is not a classical polytope. In fact, if we identify the complex space $\mathbb{C}^n$ with the real space $\mathbb{R}^{2n}$, we see that $\mathscr{P}$ is not bounded by hyperplanes. In general even the intersection $\mathscr{P} \bigcap \mathbb{R}^n$ is not a classical polytope. However, if the b.c.p. $\mathscr{P}$ admits an essential system of real vertices, then $\mathscr{P} \bigcap \mathbb{R}^n$ is a classical polytope.

Furthermore given a b.c.p. $\mathscr{P}$ the essential system of vertices $\mathscr{X}$ is uniquely determined modulo scalar factors of modulus 1 as stated in the following Proposition

5.1.7 Proposition. *Assume that* $\mathscr{X} = \{x_i\}_{i=1}^m$ *and* $\overline{\mathscr{X}} = \{\bar{x}_i\}_{i=1}^k$ *are two essential systems of vertices for a b.c.p.* $\mathscr{P}$. *Then*
$k = m$ *and, for every* $i = 1,\dots,m$, *there exist* $j_i \in [1,m]$ *and* $u_i \in \mathbb{C}$ *with* $|u_i| = 1$ *s.t.* $\bar{x}_i = u_i \cdot x_{j_i}$.

Now we extend the concept of *real polytope norm* to the complex case in a straightforward way.

5.1.8 Lemma. *Any b.c.p.* $\mathscr{P}$ *is the unit ball of a norm* $\|\cdot\|_{\mathscr{P}}$ *on* $\mathbb{C}^n$.

Proof. Suppose that $\mathscr{P} = \mathsf{absco}(\mathscr{X})$, since $\mathsf{span}(\mathscr{X}) = \mathbb{C}^n$, the set $\mathscr{P}$ is *absorbing.* Thus, since it is also absolutely convex and bounded, for every $z \in \mathbb{C}^n$ the Minkowski functional associated with $\mathscr{P}$ (see [88])

$$\|z\|_{\mathscr{P}} = \inf\{\lambda > 0 \mid z \in \lambda\mathscr{P}\}. \tag{5.8}$$

is a vector norm on $\mathbb{C}^n$. □

5.1.9 Definition (Complex polytope norm)**.** *We shall call* complex polytope norm *any norm* $\|\cdot\|_{\mathscr{P}}$ *whose unit ball is a b.c.p.* $\mathscr{P}$.

Let $\mathscr{P}$ be a b.c.p. with $\mathscr{X} = \{x_i\}_{i=1}^m$ an essential system of vertices and let $\|\cdot\|_{\mathscr{P}}$ be the corresponding complex polytope norm. Then, for any $z \in \mathbb{C}^n$, the complex polytope vector norm associated with $\mathscr{P}$ is given by

$$\|z\|_{\mathscr{P}} = \min\left\{ \sum_{i=1}^m |\lambda_i| \;\middle|\; z = \sum_{i=1}^m \lambda_i x_i \right\}, \tag{5.9}$$

This equality is obtained by rewriting (5.8) taking into account equations (5.6) and (5.7).

The next Theorem shows that the set of the complex polytope norms is *dense* in the set of all norms defined on $\mathbb{C}^n$ and that, consequently, the corresponding set of induced matrix complex polytope norms is *dense* in the set of all induced $n \times n$-matrix norms.

5.1.10 Theorem ([82]). *Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. Then, for any $\varepsilon > 0$, there exists a b.c.p. $\mathscr{P}_\varepsilon$ whose corresponding complex polytope norm $\|\cdot\|_\varepsilon$ satisfies the inequalities*

$$\|x\| \le \|x\|_\varepsilon \le (1+\varepsilon)\|x\| \quad \textit{for all } x \in \mathbb{C}^n. \tag{5.10}$$

Moreover, denoting by $\|\cdot\|$ and $\|\cdot\|_\varepsilon$ also the corresponding induced matrix norms, it holds that

$$(1+\varepsilon)^{-1}\|A\| \le \|A\|_\varepsilon \le (1+\varepsilon)\|A\| \quad \textit{for all } A \in \mathbb{C}^{n\times n}. \tag{5.11}$$

Due to such density property and recalling Definition 4.2.6 and Theorem 4.2.7 we have the following result

5.1.11 Theorem. *The spectral radius of a bounded family $\mathscr{F}$ of real $n\times n$–matrices is characterized by the equality*

$$\rho(\mathscr{F}) = \inf_{\|\cdot\| \in \mathscr{N}_{pol}} \|\mathscr{F}\| \tag{5.12}$$

where $\mathscr{N}_{pol}$ denotes the set of all possible induced $n\times n$–matrix complex polytope norms.

We recall that nondefective families are, by Proposition 4.2.12, such that the $\inf$ in (4.28) is a $\min$. The natural question that now arises is when a nondefective family admits an extremal complex polytope norm i.e. under which conditions the extremal norm associated with a nondefective family can be an extremal complex polytope norm (the $\inf$ in (5.12) becomes a $\min$).

In [76] Guglielmi, Wirth and Zennaro proposed the following conjecture

5.1.12 Conjecture (CPE Conjecture). *Assume that a finite family of complex $n\times n$-matrices $\mathscr{F} = \{A_i\}_{i=1}^m$ is nondefective and has s.m.p. $\overline{P}$. Then there exists an extremal complex polytope norm for $\mathscr{F}$.*

We recall that the opposite implication, a nondefective family has an s.m.p. if it admits an extremal complex polytope norm, it has been proved by the same authors (ref page 125) generalizing the results by Gurvits [84] and Lagarias and Wang [96].

Although counterexamples to this conjecture have been found (see [92]), such implication it has been proved to hold true under some conditions: this is the so–called *Small CPE Theorem* [76]. In order to present this result we need to introduce additional theoretical properties and definitions.

As already noticed, after choosing $Q \in \mathscr{P}_k(\mathscr{F})$, with $k \geq 1$, such that $\alpha = \rho(Q)^{1/k} > 0$, it is convenient to consider a scaling of the original family $\mathscr{F} = \{A_i\}_{i \in \mathscr{I}}$ by the scalar α

$$\mathscr{F}^* = \left\{\alpha^{-1} A_i\right\}_{i \in \mathscr{I}}. \tag{5.13}$$

In such a way, in fact, we automatically have $\rho(\mathscr{F}^*) \geq 1$. We remark that a suitable product Q can be selected using, for instance, the above mentioned Gripenberg's algorithm [74], a choice for Q is that of the product determining the lower bound $\overline{\rho}_k(\mathscr{F})$ with k sufficiently large (as mentioned in the previous Chapter luckily the length of s.m.p.'s of random families is quite short in general). Then, considering the associated multiplicative semigroup $\mathscr{P}(\mathscr{F}^*)$ as defined in equation (4.19), we have that

5.1.13 Theorem (Barabanov [51])**.** *If $\mathscr{P}(\mathscr{F}^*)$ is bounded, then $\mathscr{F}^*$ has an extremal norm and $\rho(\mathscr{F}^*) = 1$*

This implies that $\mathscr{F}$ is nondefective, it satisfies the finiteness property and Q is an s.m.p. for the family.

The following Theorem, that is a slight variant of a result proved by Protasov [106], illustrates the possible use of trajectories (ref Definition 4.2.8) to construct an extremal norm, the existence of which is guaranteed by Barabanov's Theorem.

5.1.14 Theorem (Guglielmi and Zennaro [83, Theorem 2.1])**.** *Consider a family $\mathscr{F}^*$ of $n \times n$–matrices, a vector $x \in \mathbb{C}^n$ and the associated trajectory $\mathscr{T}[\mathscr{F}^*, x]$ be such that:*

(i) $\rho(\mathscr{F}^*) \geq 1$

(ii) $\operatorname{span}(\mathscr{T}[\mathscr{F}^*, x]) = \mathbb{C}^n$*;*

(iii) *$\mathscr{T}[\mathscr{F}^*, x]$ is a bounded subset of $\mathbb{C}^n$.*

Then
$\mathscr{F}^*$ *is nondefective,* $\rho(\mathscr{F}^*) = 1$ *and* $\mathscr{S}[\mathscr{F}^*,x] = \overline{\text{absco}(\mathscr{T}[\mathscr{F}^*,x])}$ *is the unit ball of an extremal norm* $\|\cdot\|$ *for* $\mathscr{F}^*$ *(that is,* $\|\mathscr{F}^*\| = 1$*).*

Proof. By (ii) and (iii) the absolutely convex set

$$\mathscr{S} = \mathscr{S}[\mathscr{F}^*,x] = \overline{\text{absco}(\mathscr{T}[\mathscr{F}^*,x])}$$

is bounded and absorbing. This means that we can define a vector norm by means of the Minkowski functional associated with $\mathscr{S}$ (see [88]),

$$\|z\|_{\mathscr{S}} = \inf\{\lambda > 0 \mid z \in \lambda\mathscr{S}\}. \tag{5.14}$$

Now, by definition of $\mathscr{S}$,

$$A_i^*\mathscr{S} \subseteq \mathscr{S} \qquad \forall\, A_i^* \in \mathscr{F}^*$$

which means that the family $\mathscr{F}^*$ maps the set $\mathscr{S}$ into itself. Therefore by submultiplicativity of the norms and by Theorem 5.1.13

$$\|\mathscr{F}^*\|_{\mathscr{S}} \leq 1 \implies \mathscr{P}(\mathscr{F}^*) \text{ is bounded} \implies \rho(\mathscr{F}^*) = 1$$

□

When $\rho(\mathscr{F}^*) = 1$, i.e. when the candidate s.m.p. $Q \in \mathscr{P}_k(\mathscr{F})$ that we use to scale the family is actually an s.m.p. for $\mathscr{F}$, building the trajectory provides a tool for the construction of the unit ball of an extremal norm and, hence, for the computation of the spectral radius.

Assume that the hypotheses of Theorem 5.1.14 hold. The possibility of actually determining an extremal polytope norm, if any, is based on the search for a *suitable initial vector* x to which it corresponds a trajectory such that the set $\mathscr{S}[\mathscr{F}^*,x]$ is a balanced complex polytope: this is the aim of the Small CPE Theorem.

5.1.15 Definition. *Let* $\mathscr{F}$ *be a family of complex* $n \times n$*-matrices and* $\mathscr{F}^* = (1/\rho(\mathscr{F}))\mathscr{F}$ *be the corresponding normalized family. A set* $\mathscr{X} \subset \mathbb{C}^n$ *is said to be* $\mathscr{F}$-cyclic *if for any pair* $(x,y) \in \mathscr{X} \times \mathscr{X}$*, there exist* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \cdot |\beta| = 1$ *and two normalized products* $P^*, Q^* \in \mathscr{P}(\mathscr{F}^*)$ *such that*

$$y = \alpha P^* x \quad \text{and} \quad x = \beta Q^* y.$$

5.1.16 Definition. *A nondefective bounded family $\mathscr{F}$ of complex $n \times n$-matrices is said to be* asymptotically simple *if the set $\mathscr{E}$ of its leading eigenvectors (see Definition 4.2.17) is finite (modulo scalar nonzero factors) and $\mathscr{F}$-cyclic.*

5.1.17 Theorem (Small CPE Theorem [76, Theorem 5.2])**.** *Assume that a finite family $\mathscr{F}^*$ of complex $n \times n$-matrices fulfils the assumptions of Theorem 5.1.14. Furthermore, assume that*

(iv) $\mathscr{F}^*$ is asymptotically simple;

(v) x is a leading eigenvector of $\mathscr{F}^*$.

Then the set

$$\partial \mathscr{S}[\mathscr{F}^*, x] \bigcap \mathscr{T}[\mathscr{F}^*, x] \tag{5.15}$$

is finite modulo scalar factors of unitary modulus. Consequently, there exist a finite number of products $P^*_{(1)}, \ldots, P^*_{(s)} \in \mathscr{P}(\mathscr{F}^*)$ such that

$$\mathscr{S}[\mathscr{F}^*, x] = \text{absco}\left(\{x, P^*_{(1)}x, \ldots, P^*_{(s)}x\}\right), \tag{5.16}$$

so that $\mathscr{S}[\mathscr{F}^*, x]$ is a b.c.p.

The authors prove also the following refinement of Theorem 5.1.17.

5.1.18 Theorem. *Let the hypotheses of Theorem 5.1.17 hold and let $\mathscr{F}^*$ have a unique minimal s.m.p. (see Definition 4.2.17). Then all the leading eigenvectors of $\mathscr{F}^*$ (in the set $\Xi = \mathscr{E} \bigcap \partial \mathscr{S}[\mathscr{F}^*, x]$) are vertices of the b.c.p. $\mathscr{S}[\mathscr{F}^*, x]$.*

We observe that the knowledge (or the guess) of a spectrum–maximizing product is essential in order to make use of these last Theorems, but, as already mentioned, a suitable product Q can be selected using, for instance, the Gripenberg's algorithm [74], which provides candidate s.m.p.'s of progressively higher length.

Guglielmi, Wirth and Zennaro have also shown in [76, Example 6.3] that, for a different choice of the initial vector, the finite convergence may not hold. Since we are interested in finiteness properties and hence in an exact determination of an extremal norm, such a choice of the initial vector turns out to be of primary importance.

We remark that in many cases the family has a unique s.m.p. which has a unique leading eigenvalue and consequently a unique leading eigenvector. This implies that all the others leading eigenvectors, which are associated with the finitely many periodic permutations of the unique s.m.p., are trivially $\mathscr{F}$-cyclic and consequently the family is asymptotically simple.

5.1.1 A procedure for finding an extremal polytope norm

We assume that $\mathscr{F}$ is finite and nondefective. The following procedure is derived by a suitable development of previous algorithms (see[81, 83] and [106]).

The idea is that of computing iteratively the trajectory $\mathscr{T}[\mathscr{F}^*,x]$ by applying recursively the scaled family $\mathscr{F}^*$ to a suitable initial vector x. While iterating, we have to check whether $\mathscr{F}^*$ maps the convex hull of the balanced trajectory $\mathscr{T}[\mathscr{F}^*,x]$ into itself; if this holds true we stop.

5.1.19 Algorithm. *(for the construction of the unit ball of an extremal complex polytope norm for a nondefective finite family $\mathscr{F} = \{A_1,\ldots,A_\ell\}$)*
Initialization

(0) Choose a candidate s.m.p. $Q \in \mathscr{P}_k(\mathscr{F})$ (for some $k \geq 1$)

(1) Set $\rho = \rho(Q)^{1/k}$ and define the scaled family

$$\mathscr{F}^* = \{\rho^{-1}A_i\}_{i\in\{1,\ldots,\ell\}} \text{ which is such that } \rho(\mathscr{F}^*) \geq 1$$

(2) Compute the candidate leading eigenvector u of $\mathscr{F}^*$ associated with Q and set $v_0 = u$

(3) Define for step 0 the sets:

the essential system of vertices $\mathscr{W}^{(0)} = \{v_0\}$

new vectors generated $\mathscr{V}^{(0)} = \{v_0\}$

essential vertices added $\mathscr{X}^{(0)} = \{v_0\}$

the absolute convex hull $\mathscr{P}^{(0)} = \mathsf{absco}(\mathscr{W}^{(0)})$.

(4) Set $s = 1$

Main iteration

(5) Compute the set of new vectors generated at step s

$$\mathscr{V}^{(s)} = \mathscr{F}^* \left(\mathscr{X}^{(s-1)} \right).$$

(6) If $\mathscr{V}^{(s)} \subset \mathscr{P}^{(s-1)}$ then

Set $\mathscr{S}[\mathscr{F}^*, x] = \mathscr{P}^{(s-1)}$

Stop.

(7) Set $\mathscr{P}^{(s)} = \mathsf{absco}\left(\mathscr{W}^{(s-1)} \cup \mathscr{V}^{(s)}\right)$.

(8) Compute the essential system of vertices $\mathscr{W}^{(s)}$ of $\mathscr{P}^{(s)}$, such that

$$\mathscr{W}^{(s)} \subseteq \mathscr{W}^{(s-1)} \cup \mathscr{V}^{(s)}.$$

(9) Set $\mathscr{X}^{(s)} = \mathscr{V}^{(s)} \cap \mathscr{W}^{(s)}$ set of the essential vertices added at step s;

(10) Set $s = s + 1$ and Goto (5).

If the procedure halts for some s, then $\mathscr{S}[\mathscr{F}^*, x]$ is a polytope. Moreover, if the algorithm stops at step (6) and $\mathsf{span}\left(\mathscr{P}^{(s-1)}\right) = \mathsf{span}\left(\mathscr{W}^{(s-1)}\right) = \mathbb{C}^n$ then $\mathscr{S}[\mathscr{F}^*, x]$ is a b.c.p., i.e. determines the unit ball of an extremal complex polytope norm for $\mathscr{F}^*$, and Q is truly an s.m.p. for the family; otherwise $\mathscr{F}$ is reducible, i.e. $\mathsf{span}\left(\mathscr{W}^{(s-1)}\right)$ is a common invariant subspace of the family. When this last situation occurs, in order to proceed we may reduce (decompose) the family by means of the transformation matrix

$$M = \left[\begin{array}{cc} M_1 & M_2 \end{array} \right]$$

where $M_1 \in \mathbb{C}^{n \times n_1}$ provides a basis for the subspace $\mathsf{span}\left(\mathscr{W}^{(s-1)}\right)$ and $M_2 \in \mathbb{C}^{n \times n_2}$ (where $n_2 = n - n_1$) gives a basis for its complement in $\mathbb{C}^n$.

We obtain a transformed family made up of

$$M^{-1} A_i M = \left[\begin{array}{cc} A_i & B_i \\ 0 & C_i \end{array} \right] \quad \text{with} \quad i \in \{1, \ldots, \ell\} \tag{5.17}$$

which has the same joint spectral radius as the original family (ref Property 4 (4.32)). More precisely we have that

$$\rho(\mathscr{F}) = \max\{\rho(\mathscr{F}_1), \rho(\mathscr{F}_2)\},$$

where $\mathscr{F}_1 = \{A_i\}_{i=1}^{\ell}$ and $\mathscr{F}_2 = \{C_i\}_{i=1}^{\ell}$ (ref Property 6 on page 114).

If we apply the same algorithm to both family $\mathscr{F}_1$ and $\mathscr{F}_2$, iterating this process if necessary and making use also of the stopping criterion described in the following, we can get the spectral radius of the original family $\mathscr{F}$ as the maximum between $\rho(\mathscr{F}_1)$ and $\rho(\mathscr{F}_2)$. If $\rho(\mathscr{F}_1) < \rho(\mathscr{F}_2)$ [or $\rho(\mathscr{F}_1) > \rho(\mathscr{F}_2)$] we can construct an extremal polytope norm for the original family using the leading eigenvector of $\mathscr{F}_2$ [or $\mathscr{F}_1$]. If instead $\rho(\mathscr{F}_1) = \rho(\mathscr{F}_2)$ it is an open problem how to guarantee the existence and construct an extremal polytope norm for the original family $\mathscr{F}$.

We observe that the same algorithm may be applied also to defective families. Theorem 4.2.14 ensures that a defective family is always reducible (ref Fig. 4.1) so, making use of the previous algorithm, we should be able to reduce correctly the family into two families $\mathscr{F}_1$ and $\mathscr{F}_2$ and iterating this process we may be lead to identify the spectral radius of the original family. However in this case we can not guarantee the success of the algorithm. Given a candidate s.m.p. Q for the defective family and supposed x is an associated candidate leading eigenvector, we do not know a priori if x belongs or not to the nontrivial invariant subspace of the family, property that is essential: if x does not belong to the nontrivial invariant subspace of $\mathscr{F}$ and we use it as initial vector, the algorithm ends up by computing just a diverging trajectory that span the entire space even if Q is truly an s.m.p. of the family. Consider, in fact, that the defectiveness of the family does not imply in general the defectiveness of the possible s.m.p., as shown by Guglielmi and Zennaro in [79]. The authors present examples of defective 3×3 and 4×4 families that do not admit any defective s.m.p.'s (and also any limit spectrum maximizing products ref [79]). So it can happen that $\mathscr{F}$ is defective, but the candidate s.m.p. Q is nondefective and this makes very intricate to guarantee a priori that a leading eigenvector of Q belongs to the nontrivial invariant subspace of the family.

Finally we observe that, when the family is real and the leading eigenvector associated with the candidate s.m.p. Q is real as well, we can apply a simpler version of the previous algorithm: instead of considering absolute convex hulls and complex polytope norms we can deal directly with convex hulls and real polytope norms. In such a way at step (3) and (7) the equations become

$$\mathscr{P}^{(0)} = \mathrm{co}\left(\mathscr{W}^{(0)}, -\mathscr{W}^{(0)}\right)$$

$$\mathscr{P}^{(s)} = \mathrm{co}\left(\mathscr{W}^{(s-1)} \cup \mathscr{V}^{(s)} \cup -\mathscr{W}^{(s-1)} \cup -\mathscr{V}^{(s)}\right)$$

and therefore we have to solve a linear programming problem. This is always the case when we study the finiteness property of pairs of 2×2 sign–matrices.

A stopping criterion

A useful criterion to stop the iteration and eventually discard the candidate s.m.p. Q is given by the following Theorem (again, see also [106]).

5.1.20 Theorem. *Let $\mathscr{F}$ be a finite irreducible family of matrices. Then $\rho(\mathscr{F}^*) > 1$ if and only if, at some step s of Algorithm 5.1.19, v_0 lies strictly inside $\mathscr{P}^{(s)}$, that is $v_0 \in \mathring{\mathscr{P}}^{(s)}$.*

Proof. Assume that, at some step s, $v_0 \in \mathring{\mathscr{P}}^{(s)}$. This would mean that there exists $x_s \in \partial \mathscr{P}^{(s)}$ such that $x_s = \beta_s v_0$ with $|\beta_s| > 1$. Let $\mathscr{V}^{(s)} = \{v_i\}_{i=1}^m$ be an essential system of vertices of $\mathscr{P}^{(s)}$. Thus we can write

$$x_s = \sum_{i=1}^m \lambda_i v_i \quad \text{with} \quad \lambda_i{}'\mathrm{s} \in \mathbb{C} \quad \text{and} \quad \sum_{i=1}^m |\lambda_i| \leq 1. \tag{5.18}$$

Since, by construction, for each i there exists a finite product $P_i \in \mathscr{P}(\mathscr{F}^*)$ such that $v_i = P_i v_0$, there must exist at least a product $P \in \mathscr{P}(\mathscr{F}^*)$ such that $\|P v_0\|_{\mathscr{P}^{(s)}} = 1$. Using the fact that $1 = \|x_s\|_{\mathscr{P}^{(s)}} = |\beta_s| \cdot \|v_0\|_{\mathscr{P}^{(s)}}$, we have

$$\|P v_0\|_{\mathscr{P}^{(s)}} = |\beta_s| \cdot \|v_0\|_{\mathscr{P}^{(s)}} > \|v_0\|_{\mathscr{P}^{(s)}} \implies \|P\|_{\mathscr{P}^{(s)}} \geq |\beta_s| > 1. \tag{5.19}$$

Thus $\|\mathscr{F}^*\|_{\mathscr{P}^{(s)}} > 1$.

Since $\mathscr{P}^{(s)} \subseteq \mathscr{P}^{(s+1)}$, we would still have $v_0 \in \mathring{\mathscr{P}}^{(s+1)}$ and the previous condition would occur for all subsequent values of s, with $|\beta_{s+1}| \geq |\beta_s|$.

If $\rho(\mathscr{F}^*) = 1$, by the irreducibility assumption, $\mathscr{P}^{(s)}$ would converge to some centrally symmetric convex set as $s \to \infty$.

Consequently there would exist $\hat{s}$ such that $\|P\|_{\mathscr{P}^{(r)}} < |\beta_s|$ for all $r > \hat{s}$, which is not possible. Consequently $\rho(\mathscr{F}^*) > 1$. Viceversa, by the irreducibility assumption, if $\rho(\mathscr{F}^*) > 1$ then

$$\lim_{s\to\infty} \mathscr{P}^{(s)} = \mathbb{C}^n.$$

This implies that there exists s such that $v_0 \in \overset{\circ}{\mathscr{P}}{}^{(s)}$. □

5.2 Application: Finiteness property of pairs of matrices in $M_2(\mathbb{S})$

In this Section we address the problem of establishing the finiteness property of pairs of 2×2 sign–matrices. We recall that a set of matrices has the *finiteness property* if the maximal rate of growth, in the multiplicative semigroup it generates, is given by the powers of a finite product.

As a main tool of our proof we make use of the procedure, presented in the previous Section, to find a so–called *real extremal polytope norm* for the set. In particular, we will check if a certain product in the multiplicative semigroup is spectrum maximizing using the algorithm presented in the previous Section.

For pairs of sign–matrices we develop the computations exactly and hence we are able to prove analytically the finiteness property. On the other hand, the algorithm can be used in a floating point arithmetic and provide a general tool for approximating the joint spectral radius of a set of matrices.

We start denoting by $M_n(\mathbb{S})$ the set of pairs of $n\times n$ matrices with entries in $\mathbb{S} = \{-1, 0, +1\}$. We recall Conjecture 4.2.22 by Blondel, Jungers and Protasov which state that every pair of $n\times n$ sign–matrices has the finiteness property.

We consider here the case of a family $\mathscr{F} = \{A, B\}$ where $A, B \in M_2(\mathbb{S})$.

The number of ordered pairs $N_o = (3^4 - 3)(3^4 - 5) = 5928$ (obtained discarding the zero matrix, the identity and its opposite from the set and the cases where the second matrix is equal to the first one or its opposite) is very large, but the number of cases to examine is immediately reduced to

$N_e = N_o/8$, since the joint spectral radius of the sets $\{\pm A, \pm B\}$ does not change as well as it does not depend of the ordering of the two matrices. Hence $N_e = 741$, which is still a quite large number of cases. By using suitable properties, we shall see that the actual number of essential cases to examine is much lower.

Mainly, the properties we shall use are based on suitable similarity transformations, which do not change the joint spectral radius.

As in [91], in order to analyze the essential cases, we separate them into classes (n_0, n_1), where n_0 is the number of non–zero entries of A and n_1 is the number of non–zero entries of B. By symmetry, we can assume $n_0 \geq n_1$. Our approach consists in showing the finiteness property of every considered case by determining explicitly the associated s.m.p., in most cases through the construction of the unit ball of a suitable real extremal polytope norm. This does not allow a unified proof but, instead, requires to treat most of the essential cases separately.

Although all the pairs of binary matrices have already been considered in [91], here we reconsider the most difficult cases because our procedure is quite different from that used in [91] and does not rely on the possible non–negativity of the matrices.

The set of *representative* matrices (we exclude $-A$ if we consider A) with a single non–zero entry which has to be considered is given by $\mathbf{C} = \{C_i\}_{i=1}^4$ with

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The set of representative matrices with two non–zero entries which has to be considered is given by $\mathbf{D} = \{D_i\}_{i=1}^{11}$ with

$$D_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix},$$

$$D_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_6 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad D_7 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad D_8 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix},$$

$$D_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D_{11} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The set of representative matrices with three non–zero entries which has to be considered is given by $\mathbf{E} = \{E_i\}_{i=1}^{16}$ with

$$E_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad E_8 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

$$E_9 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad E_{10} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad E_{11} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$E_{13} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad E_{14} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad E_{15} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad E_{16} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

The set of representative matrices with four non–zero entries which has to be considered is given by $\mathbf{F} = \{F_i\}_{i=1}^{8}$ with

$$F_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad F_4 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$F_5 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad F_6 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad F_7 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad F_8 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now consider the similarity transformations associated with the following matrices:

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which are such that $P_1^2 = I$, $P_2^2 = I$, $P_3^2 = -I$.

Clearly, for $k = 1,2,3$ we have that

$$P_k C_i P_k^{-1} \in \pm\mathbf{C}, \quad P_k D_i P_k^{-1} \in \pm\mathbf{D}, \quad P_k E_i P_k^{-1} \in \pm\mathbf{E}, \quad P_k F_i P_k^{-1} \in \pm\mathbf{F}, \tag{5.20}$$

so that these similarities do not change the finiteness property, nor the fact that the matrices are sign–matrices.

In detail, denoting by $\sim$ a similarity relation, we get

$$D_1 \sim D_2 \sim D_3 \sim D_4, \quad D_5 \sim D_6 \sim D_7 \sim D_8, \tag{5.21}$$

$$\begin{aligned} &E_1 \sim E_4 \sim E_{13} \sim E_{16}, \quad E_2 \sim E_3 \sim E_{14} \sim E_{15}, \\ &E_5 \sim E_6 \sim E_9 \sim E_{10}, \quad E_7 \sim E_8 \sim E_{11} \sim E_{12}, \end{aligned} \tag{5.22}$$

$$F_1 \sim F_4, \quad F_2 \sim F_3, \quad F_5 \sim F_6, \quad F_7 \sim F_8. \tag{5.23}$$

As we have mentioned, in the sequel we shall denote by

$$\mathscr{F}^* = (1/\rho(P))^{1/k}\,\mathscr{F} \quad \text{for some } P \in \mathscr{P}_k(\mathscr{F}) \text{ s.t. } \rho(P) \neq 0$$

and call it the scaled family.

Our aim will be to prove that $\mathscr{F}^*$ has joint spectral radius equal to 1 (which implies that P is an s.m.p. of $\mathscr{F}$).

In several cases we shall observe that one of the following standard norms, $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$, is extremal.

Some other cases are easily treated by observing that the real polytope norm $\|\cdot\|_*^+$, associated with the b.r.p. $\mathscr{P}^+ = \mathrm{co}\,(V,-V)$ with $V = \{v_0, v_1, v_2\}$, where

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

or $\|\cdot\|_*^-$, associated with the b.r.p. $\mathscr{P}^- = \mathrm{co}\,(W,-W)$ with $W = \{w_0, w_1, w_2\}$, where

$$w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

is extremal.

For some cases we make use of the spectral radius properties previously mentioned, while all the other cases are treated by using Algorithm 5.1.19.

Before summarizing the results through different tables, we give an extensive proof of an illustrative case.

5.2.1 Illustrative case

Consider the case $A = E_2$, $B = D_{11}$.

We want to prove that $P = ABA^2B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = \left(\frac{3+\sqrt{5}}{2}\right)^{1/5}$ and a real extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = A^* v_3$, $v_5 = A^* v_4$, $v_6 = B^* v_5$. To this aim, set $\gamma = \frac{1}{\rho(P)^{1/5}} \approx 0.825$. Then we get

$$v_0 = \begin{pmatrix} 1 \\ \frac{2}{1+\sqrt{5}} \end{pmatrix}, \quad v_1 = \gamma \begin{pmatrix} \frac{2}{1+\sqrt{5}} \\ -1 \end{pmatrix}, \quad v_2 = \gamma^2 \begin{pmatrix} \frac{3+\sqrt{5}}{1+\sqrt{5}} \\ \frac{2}{1+\sqrt{5}} \end{pmatrix},$$

$$v_3 = \gamma^3 \begin{pmatrix} 1 \\ \frac{3+\sqrt{5}}{1+\sqrt{5}} \end{pmatrix}, \quad v_4 = \gamma^3 \begin{pmatrix} \frac{2}{1+\sqrt{5}} \\ -\frac{3+\sqrt{5}}{1+\sqrt{5}} \end{pmatrix}, \quad v_5 = \gamma^4 \begin{pmatrix} \frac{3+\sqrt{5}}{1+\sqrt{5}} \\ -1 \end{pmatrix}.$$

As illustrated in Figure 5.1, we analyze the transformed vectors $\mathscr{F}^*(V)$. Some of them are vertices themselves by construction of $\mathscr{P}$ and, hence, do not need to be analyzed. Here we report such vectors together with the minimizing convex combinations of vertices of $\mathscr{P}$ which determine their norms (see (5.4)):

$$\begin{aligned}
A^* v_0 &= \gamma \begin{pmatrix} \frac{\sqrt{5}-1}{1+\sqrt{5}} \\ 1 \end{pmatrix} = \lambda v_3 + \mu(-v_4), \quad \lambda = \frac{2(3+\sqrt{5})}{\gamma^2(11+5\sqrt{5})}, \ \mu = \frac{2}{\gamma^2(7+3\sqrt{5})}, \\
&\quad \|A^* v_0\|_{\mathscr{P}} = \lambda + \mu \approx 0.90; \\
A^* v_1 &= v_2; \\
A^* v_2 &= v_3; \\
A^* v_3 &= \lambda(-v_1), \qquad \lambda = \gamma^3, \quad \|A^* v_3\|_{\mathscr{P}} = \lambda \approx 0.56; \\
A^* v_4 &= \gamma^4 \begin{pmatrix} \frac{5+\sqrt{5}}{1+\sqrt{5}} \\ \frac{2}{1+\sqrt{5}} \end{pmatrix} = \lambda v_2 + \mu v_5, \quad \lambda = \frac{4(2+\sqrt{5})}{7+3\sqrt{5}} \gamma^2, \ \mu = \frac{2}{7+3\sqrt{5}}, \\
&\quad \|A^* v_4\|_{\mathscr{P}} = \lambda + \mu \approx 0.98; \\
A^* v_5 &= v_0;
\end{aligned}$$

$$
\begin{aligned}
B^*v_0 &= v_1;\\
B^*v_1 &= \lambda(-v_0), \qquad \lambda = \gamma^2, \quad \|B^*v_0\|_{\mathscr{P}} = \lambda \approx 0.68;\\
B^*v_2 &= v_4;\\
B^*v_3 &= v_5;\\
B^*v_4 &= \lambda(-v_2), \qquad \lambda = \gamma^2, \quad \|B^*v_4\|_{\mathscr{P}} = \lambda \approx 0.68;\\
B^*v_5 &= \lambda v_3, \qquad \lambda = \gamma^2, \quad \|B^*v_5\|_{\mathscr{P}} = \lambda \approx 0.68.
\end{aligned}
$$

This proves the extremality of $\|\cdot\|_{\mathscr{P}}$ and that $P = ABA^2B$ is an s.m.p.

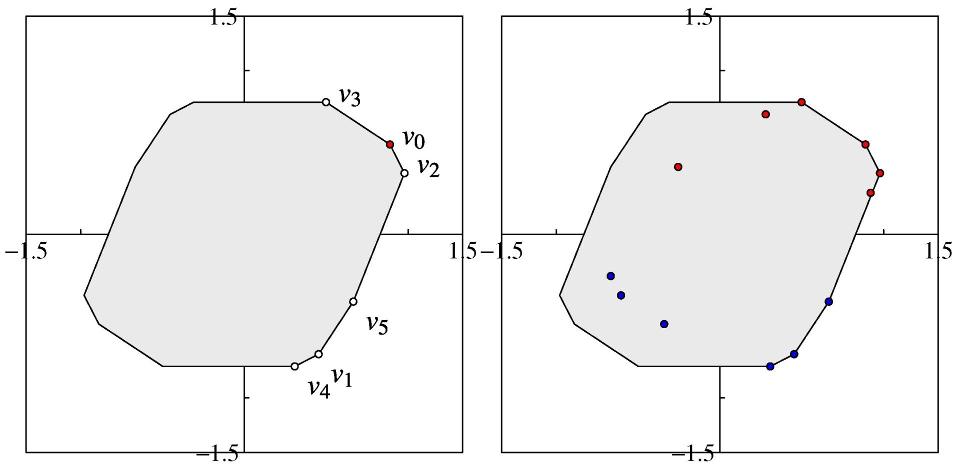


Figure 5.1: Polytope norm for the pair $\{A = E_2,\, B = D_{11}\}$ (left) and the set $\mathscr{F}^*(V)$ (right). Red points indicate the vectors $\{A^*v_i\}_{i=0}^5$ and blue points indicate the vectors $\{B^*v_i\}_{i=0}^5$.

5.2.2 Summary of results

We show in the subsequent tables the s.m.p. (s.m.p.'s) for all significant cases, that is for those matrix pairs whose analysis cannot be reduced to that of another matrix pair appearing in the tables. Rows correspond to a specific matrix A while columns to a matrix B in the pair $\mathscr{F} = \{A, B\}$.

For a detailed analysis of specific cases we refer the reader to Appendix A.

The case $n_0 = 1$ (families of the type $\mathscr{F} = \{C_i, C_j\}$).

Recall that we suppose $n_0 \geq n_1$. The only possibility is $(n_0, n_1) = (1, 1)$, corresponding to families of the type $\mathscr{F} = \{C_i, C_j\}$ $(i < j)$.

The analysis is always trivial. In fact, it is very easy to see that $\rho(\mathscr{F}) = 1$ and any among $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ is an extremal norm. Moreover, if $i = 1$ and $j = 4$ an s.m.p. is $P = C_1$ or C_4, respectively. In the case of $i = 1$ and $j = 4$ is valid also the Property 10, special case 5 on page 116. While if $(i, j) = (2, 3)$ an s.m.p. is $P = C_2C_3$ by equation (4.38) of Property 10, special case 3.

$A \backslash B$	C_2	C_3	C_4
C_1	A	A	A, B
C_2		AB	B
C_3			B

The case $n_0 = 2$.

The subcase $(n_0, n_1) = (2, 1)$ (families of the type $\mathscr{F} = \{D_i, C_j\}$). Since $\|C_j\|_1 = \|C_j\|_\infty = 1$, $\rho(D_i) = 1$ and either $\|D_i\|_1 = 1$ or $\|D_i\|_\infty = 1$, we have that $\rho(\mathscr{F}) = 1$ and that an s.m.p. is $P = D_i$.

The subcase $(n_0, n_1) = (2, 2)$ (families of the type $\mathscr{F} = \{D_i, D_j\}$). In view of (5.20) and (5.21), we can restrict the choice of the first matrix A to the set $\mathbf{D}' = \{D_1, D_5, D_9, D_{10}, D_{11}\}$ and let the choice of B be free in $\mathbf{D}$.

In the sequel we mark by an asterisk (*) or two asterisks (**) equivalent columns.

$A \backslash B$	D_2	D_3^*	D_4^*	D_5	D_6	D_7^*	D_8^*	D_9	D_{10}	D_{11}
D_1	A, B	A, B	A, B	AB	A, B	A, B	A, B	A, B	A, B	A, B
D_5	A, B	A, B	A, B		A, B	A, B	A, B	A, B	A, B	A, B
D_9	A, B	A, B	A, B		A, B	A, B	A, B		A, B	A, B
D_{10}	A, B	A, B	A, B		A, B	A, B	A, B			A, B
D_{11}	A, B	A, B	A, B		A, B	A, B	A, B			

The case $n_0 = 3$.

In view of (5.20) and (5.22), we can restrict the choice of the first matrix A to the set $\mathbf{E}' = \{E_1, E_2, E_5, E_7\}$ and let the choice of B to be free.

The subcase $(n_0, n_1) = (3, 1)$ (families of the type $\mathscr{F} = \{E_i, C_j\}$).

$A\backslash B$	C_1^*	C_2	C_3	C_4^*
E_1	A	A	A	A
E_2	A,B	A	A	A,B
E_5	A,B	A	A^4B	A,B
E_7	A,B	A	A	A,B

The subcase $(n_0, n_1) = (3, 2)$ (families of the type $\mathscr{F} = \{E_i, D_j\}$).

$A\backslash B$	D_1	D_2	D_3^*	D_4	D_5	D_6^*	D_7	D_8	D_9	D_{10}	D_{11}
E_1	A	A	A	A	A	A	A	A	A	A	A
E_2	AB	A^2B	A,B	A,B	A,B	A,B	AB	A^2B	AB	A,B	ABA^2B
E_5	A,B	A^2B	A,B	A^5B	A^2B	A,B	A^5B	A,B	A,B	A^3B	A^4B
E_7	A,B	A,B	A,B	AB	AB	A,B	A,B	A,B	A,B	A,B	AB

The subcase $(n_0, n_1) = (3, 3)$ (families of the type $\mathscr{F} = \{E_i, E_j\}$).

$A\backslash B$	E_2	E_3	E_4^*	E_5	E_6	E_7	E_8	E_9
E_1	A	A	A,B	A	A	A	A	A
E_2		AB	B	AB^3	A^2B^3	AB	A,B	A^2B^3
E_5		A^3B^2	B		A,B	A,B	A,B	AB
E_7		A,B	B		A,B		A,B	AB^3

$A\backslash B$	E_{10}	E_{11}	E_{12}	E_{13}^*	E_{14}	E_{15}	E_{16}^*
E_1	A	A	A	A,B	A	A	A,B
E_2	AB^3	A,B	AB	B	$ABA(AB)^2B$	A,B	B
E_5	A^4B^4	A^3B	A^3B	B	A^3B	A^3B^2	B
E_7	AB^3	AB	A,B	B	A,B	AB	B

We remark that in this case we find the longest spectrum–maximizing products, of length $\ell = 8$, namely for $\mathscr{F} = \{E_5, E_{10}\}$, where $P = E_5^4 E_{10}^4$ and for $\mathscr{F} = \{E_2, E_{14}\}$, where $P = E_2 E_{14} E_2 (E_2 E_{14})^2 E_{14}$.

The case $n_0 = 4$.

In view of (5.20) and (5.23), we can restrict the choice of the first matrix A to the set $\mathbf{F}' = \{F_1, F_3, F_5, F_8\}$ and let the choice of B be free.

The subcase $(n_0, n_1) = (4, 1)$ (families of the type $\mathscr{F} = \{F_i, C_j\}$).

$A \backslash B$	C_1^*	C_2^{**}	C_3^{**}	C_4^*
F_1	A	A	A	A
F_3	A	A	A	A
F_5	B	AB	AB	B
F_8	A	A	A	A

The subcase $(n_0, n_1) = (4, 2)$ (families of the type $\mathscr{F} = \{F_i, D_j\}$).

$A \backslash B$	D_1^*	D_2^*	D_3^{**}	D_4^{**}	D_5^{**}	D_6^{**}	D_7^*	D_8^*	D_9^{**}	D_{10}^*	D_{11}^{**}
F_1	A	A	A	A	A	A	A	A	A	A	A
F_3	A	A	A	A	A	A	A	A	A	A	A
F_5	B	B	AB	AB	AB	AB	B	B	AB	B	AB
F_8	A	A	A	A	A	A	A	A	A	A	A

The subcase $(n_0, n_1) = (4, 3)$ (families of the type $\mathscr{F} = \{F_i, E_j\}$). It is useful observing that $P_3 F_1 P_3^{-1} = -F_1$, $P_3 F_3 P_3^{-1} = F_3$, $P_1 F_5 P_1^{-1} = -F_5$ and that both the similarity transformations associated with P_1 and P_3 are one–to–one applications between the sets of matrices $\mathbf{E}'' = \{E_j \mid 1 \le j \le 8\}$ and $\mathbf{E}''' = \{E_j \mid 9 \le j \le 16\}$. Consequently, when $A = F_i$ $(i = 1, 3, 5)$, we can restrict the choice of the matrix B within the set $\mathbf{E}''$.

$A \backslash B$	E_1^*	E_2	E_3	E_4^*	E_5	E_6	E_7	E_8	$E_9 - E_{16}$
F_1	B	A	A	B	A	A	A	AB	
F_3	B	$(AB)^2A^2B$	A^2BA^3B	B	A^3B^2	AB^2	A^2B	A^2B	
F_5	B	AB	B	B	AB^4	AB^4	B	AB	
F_8	A	A	A	A	A	A	A	A	A

The subcase $(n_0, n_1) = (4, 4)$ **(families of the type** $\mathscr{F} = \{F_i, F_j\}$**).** If $A = F_8$, $B \in \mathbf{F}$ then A is an s.m.p.

Now it is useful to observe that $P_3F_1P_3^{-1} = -F_1$, $P_3F_3P_3^{-1} = F_3$, $P_3F_5P_3^{-1} = -F_6$ and $P_3F_8P_3^{-1} = F_7$. Consequently, when $A = F_i$ $(i = 1, 3)$, we can restrict the choice of the matrix B within the set $\mathbf{F}'' = \{F_2, F_3, F_4, F_5\}$.

$A \backslash B$	F_2	F_3	F_4	F_5	F_6
F_1	A, B	A, B	A, B	AB	
F_3	A, B		A, B	A^2B	
F_5	AB^2		AB		AB
F_8	A	A	A	A	A

5.3 Conclusions and future work

We have proved the finiteness property for any pair of 2×2 sign–matrices. In most non–trivial cases, this has been made possible by detecting an extremal real polytope norm for the family constituted by two sign–matrices. The finite convergence of the procedure for constructing the unit ball of such a norm, carried out on a case–by–case basis, implies the finiteness property. This is a promising first step toward the validation of Conjecture 4.2.22 which would imply that finite sets of rational matrices fulfil the finiteness property (ref Theorem 4.2.20).

A Matlab version of Algorithm 5.1.19 is available on the webpage of Nicola Guglielmi[1].

Unfortunately, it seems clear that such an approach can hardly be extended to the general case of a pair of sign–matrices of arbitrary dimension.

[1] `http://univaq.it/~guglielm`

The use of an induction argument on the dimension seems difficult. Nevertheless, we plan to explore it in future.

The algorithm presented, which proves to be extremely useful in practice as demonstrated by results reported in this Chapter and in Appendix A, makes use of Theorem 5.1.17 (the so–called *Small CPE Theorem*) and its refinement Theorem 5.1.18. It would be undoubtedly useful to extend these Theorems to other cases. The task is tough (recall that Jungers and Protasov [92] have founded counterexamples to Conjecture 5.1.12), but the possible benefits are enticing.

Another problem we would like to address in a future work is that of the joint spectral radius approximation. In many practical problems it is just required to find a good approximation of the spectral radius of a family and, as we said, the Gripenberg's algorithm [74] allows to efficiently compute lower and upper bounds of $\rho(\mathscr{F})$ based on the four members inequality (5.1). However, while the lower bound $\overline{\rho}_k(\mathscr{F})$ converges usually in a few steps k to the actual value $\rho(\mathscr{F})$ due to the existence of a short s.m.p. for $\mathscr{F}$, the convergence of the upper bound $\widehat{\rho}_k(\mathscr{F})$ depends heavily on the operator norm chosen. The idea we would like to develop in a future work is that of using polytope norms as well as ellipsoidal norms [57] in the quest for tighter upper bounds.

Finally we believe that Conjecture 4.2.18, concerning the measure of the set of counterexamples to the finiteness property, deserves to be analyzed in more details and we plan to study it in a future work.

Conclusions

In the present thesis we studied spectral properties of families of matrices, with particular emphasis on rank–one perturbed matrices, stochastic matrices and generic families.

In the first Chapter we introduced the complete principle of biorthogonality, generalizing the well known Brauer's principle of biorthogonality, and we applied it to the study of the rank–one perturbed matrix $A(c) = cA + (1-c)\lambda x v^*$, with A a square complex matrix, $c \in \mathbb{C}$, x and v nonzero complex vectors such that $Ax = \lambda x$ and $v^* x = 1$. From this analysis we derived the eigenvalues and Jordan blocks of $A(c)$, an explicit expression for both the left λ–eigenvector $y(c)$ and its limit $\lim_{c \to 1} y(c)$.

In Chapter 2 first we discussed the Google PageRank model and its adherence to the reality, pointing out pathologies and limitations of the actual model and proposing some possible improvements that led us to introduce what we defined as a VisibilityRank or CommercialRank, quantity which could be of interest, besides the Web ranking, also in other applications like political/social sciences or in ranking the importance of a paper and/or of a researcher looking in scientific databases.

Then we analyzed the characteristics of the matrix $G(c) = cG + (1-c)ev^T$ as a function of the complex parameter c, with G the basic (stochastic) Google matrix and v a complex vector such that $v^* e = 1$. In particular, making use of the general matrix–theoretic analysis made in the first Chapter, we obtained the eigenvalues, Jordan form and the eigenvector structure of $G(c)$ for $c \in \mathbb{C}$. For the left 1–eigenvector of $G(c)$, which is the complex analog of the *PageRank* vector $y(c)$, we studied its regularity, limits, expansions, and conditioning as a function of c, discovering that the vector function $y(c)$ is analytic in both the disk $\{c : |1-c| < \varepsilon\}$ and the unit disk $\{c : |c| < 1\}$. Therefore, the $\lim_{c\to 1} y(c)$ exists unique and is equal to $\widetilde{y}$ regardless of the path taken, which can be a non–real path in the complex plane. We discussed about the limit value $\tilde{y} = YX^{*}v$, where v is the personalization vector and $YX^{*} = N$ is the ergodic projector coinciding with the Cesaro mean $\lim_{r\to\infty} \frac{1}{r+1}\sum_{j=0}^{r} G^{j}(1)$.

In the last part of the Chapter, resting on the analysis of the general parametric Google matrix developed in the first part, we proposed two algorithms for the efficient evaluation of the PageRank. Both of them appear to be promising, especially in the case of c close or equal to 1, and they are a preliminary step that, in our opinion, merits further research.

In Chapter 3 we presented the Vicsek model, which allows to describe the dynamics of groups of autonomous agents and the attainment of a global consensus among the agents. We observed that whether the topology of the network of agents is fixed or it varies over time, the convergence of the system to a global consensus can be traced back to the spectral properties of the matrices associated with the system which are, by construction, stochastic matrices. In the latter case checking if the second spectral radius of the family is strictly less than 1 proves to be a sufficient condition which guarantees the system to reach a global consensus $e\theta_{cons}$ independently of the initial conditions $\theta(0)$. The common value θ_{cons} depends only on the initial state of the system and on a ranking of the agents which proved to be a generalization of the Google PageRank in a dynamical context.

We believe that analysing the influence of an agent, or group of agents, on the final value θ_{cons} from a dynamical point of view may give new hints in the study of many complex processes connected to human and biological behavior like infectious diseases diffusion, goods and services prices in economy, market trends in finance, viral marketing etc.

In the last part of the thesis we studied spectral properties of generic families. In particular we presented the generalization of spectral radius to a set of square matrices, the so–called joint spectral radius, and its properties. We described an algorithm for the exact computation of this quantity which is based on the construction of extremal polytope norm for the family. We used it to prove the finiteness property for any pair of 2×2 sign–matrices. This is a promising first step toward the validation of Conjecture 4.2.22 which would imply that finite sets of rational matrices fulfil the finiteness property, prospective extremely enticing from a practical point of view.

Finally, about the approximation of the joint spectral radius, even though many algorithms have been proposed in the literature that allows to obtain good approximations, we believe that more can be done in terms of accuracy of the solution and time needed to evaluate it. For instance, the Gripenberg's algorithm [74] allows to compute lower and upper bounds of $\rho(\mathscr{F})$ based on the four members inequality (5.1) and proves to be very useful in many application. Nevertheless the achievable precision is tied to the matrix norm used which is usually far away from being the extremal one. We believe that, making use of polytope norms as well as ellipsoidal norms [57], it must be possible to increase consistently the precision of this kind of algorithms.

Appendices

Appendix A

Detailed analysis for pairs of matrices in $M_2(\mathbb{S})$

In this Appendix we provide a case–by–case analysis of the matrix pairs tabulated in Section 5.2.2. In particular we provide explicitly the computed extremal polytope norm in those cases where they have been used to determine an s.m.p.

The case $n_0 = 2$

The subcase $(n_0, n_1) = (2, 2)$ (families of the type $\mathscr{F} = \{D_i, D_j\}$).

- $A = D_1$ and $B = D_j$ $(j = 2, 3, 4, 9, 10, 11)$.

  Since $\rho(A) = \rho(B) = \|A\|_1 = \|B\|_1 = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.'s.

- $A = D_1$ and $B = D_5$.

  We have that $B = A^*$, equation (4.38) of Property 10, special case 3, ensures that $P = AB$ is an s.m.p.

- $A = D_1$ and $B = D_6$.

  We find that $P = B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = 1$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$.

- $A = D_1$ and $B = D_j$ $(j = 7,8)$.

  Since $A^2 = A$, $B^2 = B$, $\rho(AB) = \rho(BA) = 0$ and $\rho(A) = \rho(B) = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.'s.

- $A = D_5$ and $B \in \mathbf{D}$.

  Since $D_5 = D_1^T$ and $\mathbf{D}^T \subseteq \pm\mathbf{D}$ and since, if P is an s.m.p. of the family $\mathscr{F} = \{A,B\}$, then P^T is an s.m.p. of the family $\mathscr{F}^T = \{A^T, B^T\}$, we are led again to the previous cases.

- $A = D_j$ and $B = D_k$ $(j = 9,10,11,\ k = 2,3,4,6,7,8)$.

  Since $P_1D_9P_1^{-1} = -D_9$, $P_2D_9P_2^{-1} = D_9$, $P_3D_9P_3^{-1} = -D_9$, $P_1D_{10}P_1^{-1} = D_{10}$, $P_2D_{10}P_2^{-1} = -D_{10}$, $P_3D_{10}P_3^{-1} = -D_{10}$, $P_1D_{11}P_1^{-1} = -D_{11}$, $P_2D_{11}P_2^{-1} = -D_{11}$, $P_3D_{11}P_3^{-1} = D_{11}$ and since $P_1D_2P_1^{-n1} = D_1$, $P_2D_3P_2^{-1} = D_1$, $P_3D_4P_3^{-1} = -D_1$, $P_1D_6P_1^{-1} = D_5$, $P_2D_7P_2^{-1} = D_5$, $P_3D_8P_3^{-1} = -D_5$, by using the similarity transformations associated with P_1, P_2 and P_3 we are led to the previous cases.

- $A = D_j$ and $B = D_k$ $(j,k = 9,10,11)$.

  Since $\rho(A) = \rho(B) = \|A\|_\infty = \|B\|_\infty = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.'s.

The case $n_0 = 3$

The subcase $(n_0, n_1) = (3,1)$ (families of the type $\mathscr{F} = \{E_i, C_j\}$).

- $A = E_1$, $B \in \mathbf{C}$.

  We find that $P = A$ is an s.m.p., $\rho(A) = \frac{1+\sqrt{5}}{2} = \|A\|_2$ and $\|B\|_2 = \sqrt{2}$ as a result $\rho(\mathscr{F}) = \rho(A)$.

- $A = E_2$, $B \in \mathbf{C}$.

  Since $\rho(A) = \|A\|_*^+ = \|B\|_*^+ = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A is an s.m.p.

- $A = E_5$, $B = C_j$ $(j = 1,2,4)$.

  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and A is an s.m.p.

- $A = E_5$, $B = C_3$.

  We find that $P = A^4B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = 4^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0$, $v_2 = B^*v_0$, $v_3 = A^*v_2$, $v_4 = A^*v_3$, $v_5 = A^*v_4$.

- $A = E_7$, $B \in \mathbf{C}$.

  Since $\rho(A) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A is an s.m.p. In the case of $B = C_2$ also l^2 norm is extremal.

The subcase $(n_0, n_1) = (3, 2)$ (families of the type $\mathscr{F} = \{E_i, D_j\}$).

- $A = E_1$, $B \in \mathbf{D}$.

  Since $\rho(A) = \|A\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|B\|_2 \leq \sqrt{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that A is an s.m.p.

- $A = E_2$, $B = D_1$.

  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^*v_0$, $v_2 = A^*v_0$.

- $A = E_2$, $B = D_j$ $(j = 2, 8)$.

  We find that $P = A^2B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 2^{1/3}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^*v_0$, $v_2 = A^*v_1$.

- $A = E_2$, $B = D_j$ $(j = 3, 4, 5, 6, 10)$.

  Since $\rho(A) = \rho(B) = \|A\|_*^+ = \|B\|_*^+ = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.'s.

- $A = E_2$, $B = D_7$.

  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^*v_0$, $v_2 = A^*v_1$.

- $A = E_2$, $B = D_9$.

  The l^2 norm is extremal and $P = AB$ is an s.m.p., $\rho(AB) = \|AB\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|AB\|_2, \|AB\|_2, \|AB\|_2 \leq \frac{1+\sqrt{5}}{2}$ so we have $\rho(\mathscr{F}) = \rho(AB)^{1/2} = \left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$.

- $A = E_2$, $B = D_{11}$.

  See the illustrative example in Section 5.2.1.

- $A = E_5$, $B = D_j$ $(j = 1,3,6,8,9)$.

  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and both A and B are s.m.p.'s.

- $A = E_5$, $B = D_j$ $(j = 2,5)$.

  We find that $P = A^2B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 3^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0$, $v_2 = B^*v_0$, $v_3 = A^*v_1$, $v_4 = A^*v_2$.

- $A = E_5$, $B = D_j$ $(j = 4,7)$.

  We find that $P = A^5B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/6} = 2^{1/3}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0$, $v_2 = B^*v_0$, $v_3 = A^*v_2$, $v_4 = A^*v_3$, $v_5 = A^*v_4$, $v_6 = A^*v_5$.

- $A = E_5$, $B = D_{10}$.

  We find that $P = A^3B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0$, $v_2 = B^*v_0$, $v_3 = A^*v_2$, $v_4 = B^*v_1$, $v_5 = A^*v_3$.

- $A = E_5$, $B = D_{11}$.

  See the illustrative example in Section 5.2.2. We find that $P = A^4B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = (2+\sqrt{3})^{1/5}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0$, $v_2 = B^*v_0$, $v_3 = A^*v_2$, $v_4 = A^*v_3$, $v_5 = A^*v_4$, $v_6 = B^*v_5$.

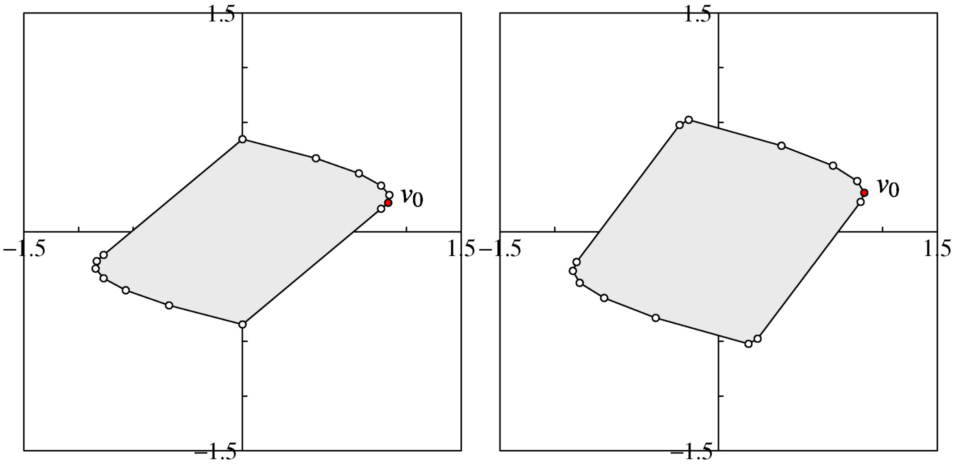


Figure A.1: Polytope norm for the pairs $\{A = E_5, B = D_4\}$ (left) and $\{A = E_5, B = D_{11}\}$ (right).

- $A = E_7$, $B = D_j$ $(j = 1,3,6,8,9)$.

  The family $\mathscr{F}$ is upper triangular with $\rho(\mathscr{F}) = 1$ and A and B are both s.m.p.'s.

- $A = E_7$, $B = D_j$ $(j = 2,7,10)$.

  Since $\rho(A) = \rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.'s.

- $A = E_7$, $B = D_j$ $(j = 4,5)$.

  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.

- $A = E_7$, $B = D_{11}$.

  We find that $P = AB$ is an s.m.p. and l^2 norm is extremal, $\rho(AB) = \|AB\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|AB\|_2, \|AB\|_2, \|AB\|_2 \leq \frac{1+\sqrt{5}}{2}$ so we have $\rho(\mathscr{F}) = \rho(AB)^{1/2} = \left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$.

The subcase $(n_0, n_1) = (3,3)$ (families of the type $\mathscr{F} = \{E_i, E_j\}$).

- $A = E_1$, $B \in \mathbf{E}$.

Since $\rho(A) = \|A\|_2 = \|B\|_2 = \frac{1+\sqrt{5}}{2}$, we have $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and A is an s.m.p., for $B = E_j$ $(j = 4, 13, 16)$ also B is an s.m.p.

- $A = E_2$, $B = E_3$.

  Using the Property 10, special case 6 on page 116, or equation (4.38) of Property 10, special case 3, we find that $P = AB$ is an s.m.p. and $\rho(\mathscr{F}) = \rho(P)^{1/2} = \frac{1+\sqrt{5}}{2}$.

- $A = E_2$, $B = E_j$ $(j = 4, 13, 16)$. Since $\rho(B) = \|A\|_2 = \|B\|_2 = \frac{1+\sqrt{5}}{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p.

- $A = E_2$, $B = E_j$ $(j = 5, 10)$.

  We find that $P = AB^3$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = (2+\sqrt{3})^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^*$ $v_4 = B^* v_2$, $v_5 = A^* v_4$, $v_6 = B^* v_4$, $v_7 = B^* v_6$.

- $A = E_2$, $B = E_j$ $(j = 6, 9)$.

  We find that $P = A^2 B^3$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = (2+\sqrt{3})^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1$, $v_3 = B^* v_2$, $v_4 = A^* v_3$, $v_5 = B^* v_3$, $v_6 = A^* v_5$.

- $A = E_2$, $B = E_j$ $(j = 7, 12)$.

  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = (1+\sqrt{2})^{1/2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$.

- $A = E_2$, $B = E_j$ $(j = 8, 11, 15)$.

  Since $\rho(A) = \rho(B) = \|A\|_*^+ = \|B\|_*^+ = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.'s. For $B = E_{15}$ holds true also the Property 10, special case 5 on page 116.

- $A = E_2$, $B = E_{14}$.

We find that $P = ABA^2BAB^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/8} = (7+4\sqrt{3})^{1/8}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = B^*v_0$, $v_2 = B^*v_1$, $v_3 = A^*v_2$, $v_4 = B^*v_3$, $v_5 = A^*v_4$, $v_6 = A^*v_5$, $v_7 = B^*v_6$.

Observe that this is the first of the two cases with the largest number of factors in the s.m.p. The essential vertices of $\mathscr{P}$ are just the leading eigenvectors of $\mathscr{F}$, that is, the eigenvectors of all the cyclic permutations of P.

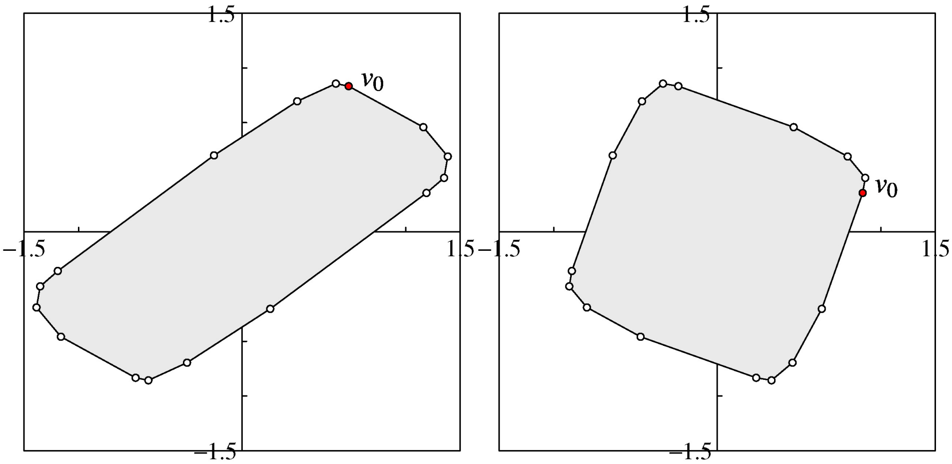


Figure A.2: Polytope norm for the pairs $\{A = E_2, B = E_5\}$ (left) and $\{A = E_2, B = E_{10}\}$ (right).

- $A = E_5$, $B = E_j$ $(j = 3, 15)$.

  We find that $P = A^3B^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = \left(2+\sqrt{3}\right)^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0$, $v_2 = B^*v_0$, $v_3 = B^*v_1$, $v_4 = B^*v_2$, $v_5 = A^*v_4$, $v_6 = A^*v_5$.

- $A = E_5$, $B = E_j$ $(j = 4, 13, 16)$.

  Since $\rho(B) = \|A\|_2 = \|B\|_2 = \frac{1+\sqrt{5}}{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p.

- $A = E_5$, $B = E_j$ $(j = 6, 7, 8)$.

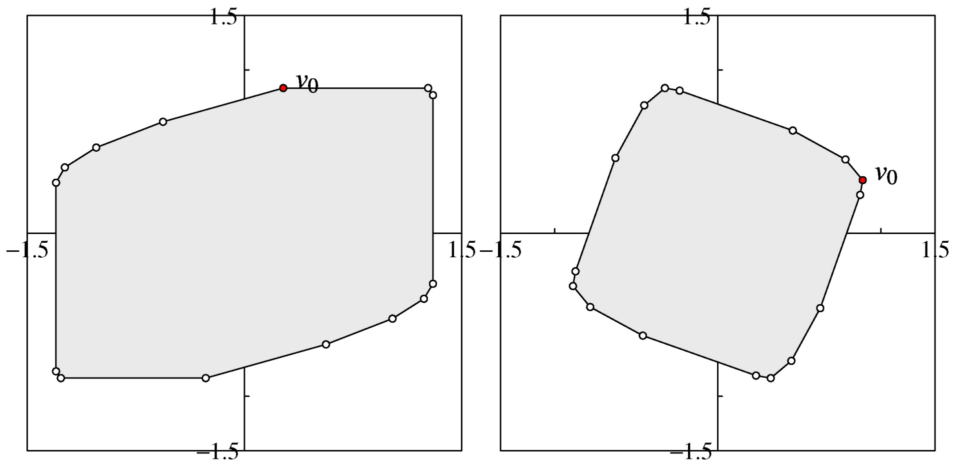


Figure A.3: Polytope norm for the pairs $\{A = E_2, B = E_6\}$ (left) and $\{A = E_2, B = E_{14}\}$ (right).

The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and both A and B are s.m.p's.

- $A = E_5$, $B = E_9$.

  We have that $B = A^*$ and, by equation (4.38) of Property 10, special case 3, $P = AB$ is an s.m.p.

- $A = E_5$, $B = E_{10}$.

  We find that $P = A^4B^4$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/8} = \left(7+4\sqrt{3}\right)^{1/8}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = B^*v_0$, $v_2 = B^*v_1$, $v_3 = B^*v_2$, $v_4 = B^*v_3$, $v_5 = A^*v_4$, $v_6 = A^*v_5$, $v_7 = A^*v_6$.

  This is the second of the two cases with the largest number of factors in the s.m.p. Again, the essential vertices of $\mathscr{P}$ are just the leading eigenvectors of $\mathscr{F}$.

- $A = E_5$, $B = E_j$ $(j = 11, 12)$.

  We find that $P = A^3B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0$, $v_2 = B^*v_0$, $v_3 = B^*v_1$, $v_4 = A^*v_2$, $v_5 = A^*v_4$.

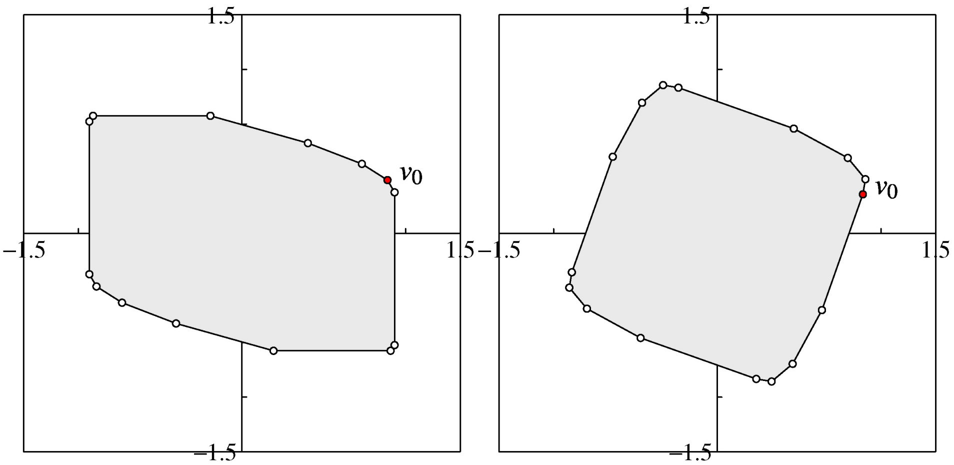


Figure A.4: Polytope norm for the pairs $\{A = E_5, B = E_3\}$ (left) and $\{A = E_5, B = E_{10}\}$ (right).

- $A = E_5,\ B = E_{14}$.

  We find that $P = A^3B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = \left(2+\sqrt{3}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0,\ v_2 = B^*v_0,\ v_3 = A^*v_2,\ v_4 = B^*v_2,\ v_5 = A^*v_3,\ v_6 = B^*v_3,\ v_7 = B^*v_5$.

- $A = E_7,\ B = E_j\ (j = 3, 12, 14)$.

  Since $\rho(A) = \rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.'s. For $B = E_{12}$ also the Property 10, special case 5 on page 116, is valid.

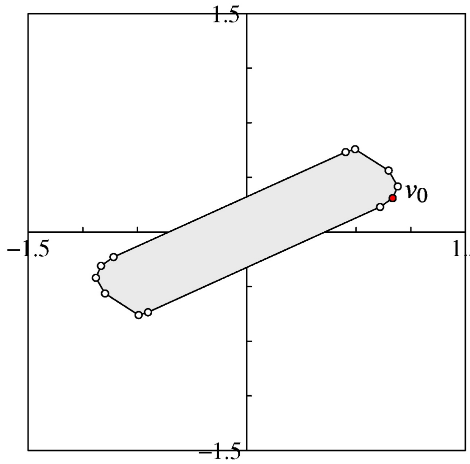
- $A = E_7,\ B = E_j\ (j = 4, 13, 16)$.

  Since $\rho(B) = \|A\|_2 = \|B\|_2 = \frac{1+\sqrt{5}}{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p.

- $A = E_7,\ B = E_j\ (j = 6, 8)$.

  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and both A and B are s.m.p's.

- $A = E_7,\ B = E_j\ (j = 9, 10)$.



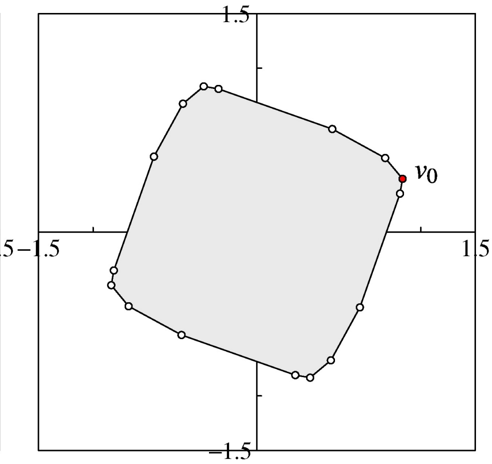


Figure A.5: Polytope norm for the pairs $\{A = E_5, B = E_{11}\}$ (left) and $\{A = E_5, B = E_{14}\}$ (right).

We find that $P = AB^3$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1$, $v_3 = B^* v_2$, $v_4 = B^* v_3$, $v_5 = A^* v_4$.

- $A = E_7$, $B = E_{11}$.

  Using the Property 10, special case 6 on page 116, we find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \frac{1+\sqrt{5}}{2}$. The same result is obtained making use of l^2 norm that proves to be extremal for this family.

- $A = E_7$, $B = E_{15}$.

  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \left(1 + \sqrt{2}\right)^{1/2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1$.

The case $n_0 = 4$

The subcase $(n_0, n_1) = (4, 1)$ (families of the type $\mathscr{F} = \{F_i, C_j\}$).

- $A = F_i$ $(i = 1, 3)$, $B \in \mathbf{C}$.

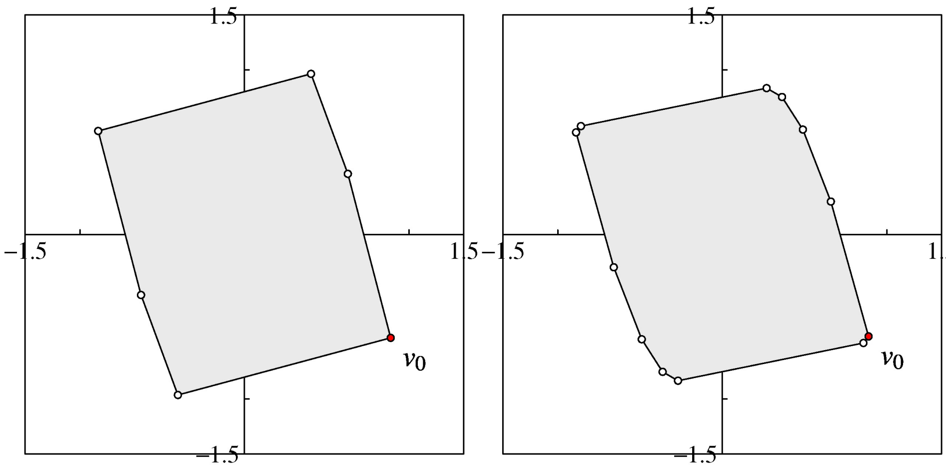


Figure A.6: Polytope norm for the pairs $\{A = E_7, B = E_{15}\}$ (left) and $\{A = E_7, B = E_9\}$ (right).

Since $\rho(A) = \|A\|_2 = \sqrt{2}$ and $\|B\|_2 = 1$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that A is an s.m.p.

- $A = F_5$, $B = C_j$ $(j = 1,4)$.

  Since $\rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that B is an s.m.p.

- $A = F_5$, $B = C_j$ $(j = 2,3)$.

  Since $\rho(AB) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that $P = AB$ is an s.m.p.

- $A = F_8$, $B \in \mathbf{C}$.

  Since $\rho(A) = \|A\|_1 = 2$ and $\|B\|_1 = 1$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p.

The subcase $(n_0, n_1) = (4,2)$ (families of the type $\mathscr{F} = \{F_i, D_j\}$).

- $A = F_i$ $(i = 1,3)$, $B \in \mathbf{D}$.

  Since $\rho(A) = \|A\|_2 = \sqrt{2}$ and $\|B\|_2 \leq \sqrt{2}$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that A is an s.m.p.

- $A = F_5$, $B = D_j$ $(j = 1,2,7,8,10)$.

Since $\rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that B is an s.m.p.

- $A = F_5,\ B = D_j\ (j = 3,4,5,6,9,11)$.

  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V,-V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$. For $B = D_j$ $(j = 3,4,9,11)$ also l^1 norm is extremal.

- $A = F_8,\ B \in \mathbf{D}$.

  Since $\rho(A) = \|A\|_2 = 2$ and $\|B\|_2 \leq \sqrt{2}$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p.

The subcase $(n_0, n_1) = (4,3)$ (families of the type $\mathscr{F} = \{F_i, E_j\}$).

- $A = F_1,\ B = E_j\ (j = 1,4)$.

  Since $\rho(B) = \|B\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|A\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p.

- $A = F_1,\ B = E_j\ (j = 2,3,7)$.

  We find that $P = A$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V,-V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$.

- $A = F_1,\ B = E_j\ (j = 5,6)$.

  We find that $P = A$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V,-V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$, $v_3 = B^* v_1$, $v_4 = A^* v_3$, $v_5 = B^* v_3$, $v_6 = A^* v_5$.

- $A = F_1,\ B = E_8$.

  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V,-V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.

- $A = F_3,\ B = E_j\ (j = 1, 4)$.

  Since $\rho(B) = \|B\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|A\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p.

- $A = F_3,\ B = E_2$.

  We find that $P = (AB)^2 A^2 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/7} = \left(4\left(2+\sqrt{3}\right)\right)^{1/7}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4,\ v_5, v_6, v_7, v_8, v_9, v_{10}\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0,\ v_2 = B^* v_0,\ v_3 = A^* v_1,\ v_4 = A^* v_2,\ v_5 = A^* v_4,\ v_6 = A^* v_5,\ v_7 = B^* v_5,\ v_8 = A^* v_7,\ v_9 = A^* v_8,\ v_{10} = B^* v_9$.

- $A = F_3,\ B = E_3$.

  We find that $P = A^2 B A^3 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/7} = \left(4\left(2+\sqrt{2}\right)\right)^{1/7}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4,\ v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0,\ v_2 = B^* v_0,\ v_3 = A^* v_2,\ v_4 = A^* v_3,\ v_5 = A^* v_4,\ v_6 = B^* v_5,\ v_7 = A^* v_6$.

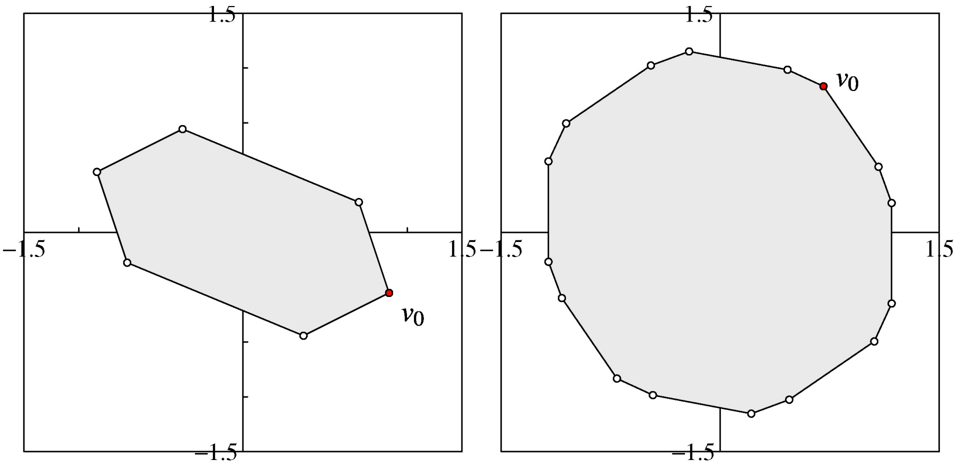


Figure A.7: Polytope norm for the pairs $\{A = F_1, B = E_3\}$ (left) and $\{A = F_3, B = E_3\}$ (right).

- $A = F_3,\ B = E_5$.

  We find that $P = A^3 B^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = \left(2\left(2+\sqrt{2}\right)\right)^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V =$

$\{v_0, v_1, v_2, v_3, v_4, \ v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$, $v_3 = B^* v_1$, $v_4 = A^* v_2$, $v_5 = A^* v_3$, $v_6 = A^* v_4$, $v_7 = A^* v_5$.

- $A = F_3$, $B = E_6$.

  We find that $P = AB^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = \left(2+\sqrt{2}\right)^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, \ v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_1$, $v_4 = A^* v_2$, $v_5 = B^* v_2$, $v_6 = A^* v_4$, $v_7 = A^* v_6$.

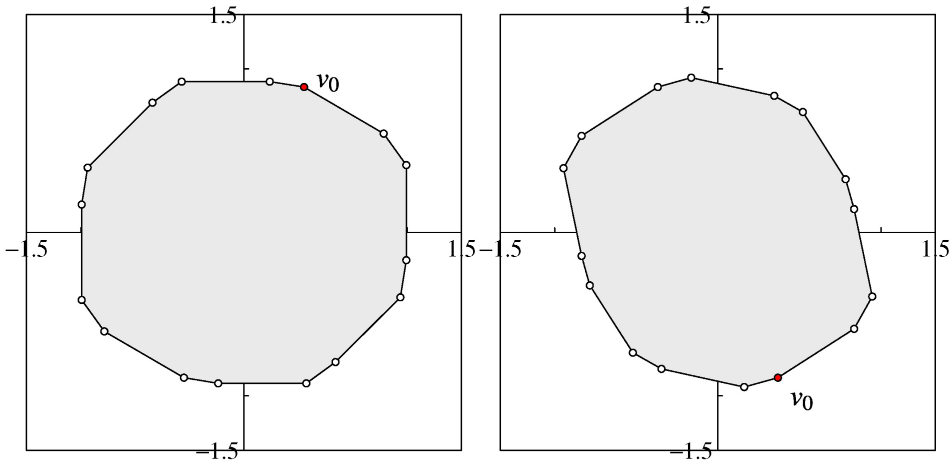


Figure A.8: Polytope norm for the pairs $\{A = F_3, B = E_5\}$ (left) and $\{A = F_3, B = E_6\}$ (right).

- $A = F_3$, $B = E_j$ $(j = 7, 8)$.

  We have that l^2 norm is extremal since $\overline{\rho}_3(\mathscr{F}) = \rho(A^2 B)^{1/3} = \widehat{\rho}_3(\mathscr{F}) = \|A^2 B\|_2^{1/3}$. $P = A^2 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = \left(1+\sqrt{5}\right)^{1/3}$.

- $A = F_5$, $B = E_j$ $(j = 1, 4)$.

  We find that $P = B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \frac{1+\sqrt{5}}{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = A^* v_0$.

- $A = F_5$, $B = E_j$ $(j = 2, 8)$.

  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.

- $A = F_5$, $B = E_j$ $(j = 3, 7)$.

  Since $\rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that B is an s.m.p.

- $A = F_5$, $B = E_j$ $(j = 5, 6)$.

  We find that $P = AB^4$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = 4^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1$, $v_3 = B^* v_2$, $v_4 = B^* v_3$, $v_5 = B^* v_4$.

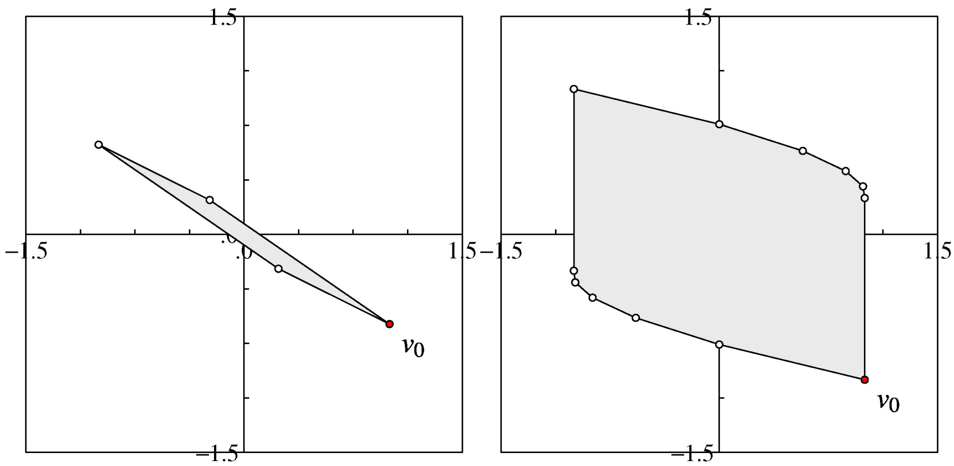


Figure A.9: Polytope norm for the pairs $\{A = F_5, B = E_4\}$ (left) and $\{A = F_5, B = E_5\}$ (right).

- $A = F_8$, $B \in \mathbf{E}$.

  Since $\rho(A) = \|A\|_1 = \|B\|_1 = 2$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p.

The subcase $(n_0, n_1) = (4, 4)$ (families of the type $\mathscr{F} = \{F_i, F_j\}$).

- $A = F_1$, $B = F_j$ $(j = 2, 3, 4)$.

  Since $\rho(A) = \rho(B) = \|A\|_2 = \|B\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that both A and B are s.m.p.'s.

- $A = F_1$, $B = F_5$. (See Figure A.10)

  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.

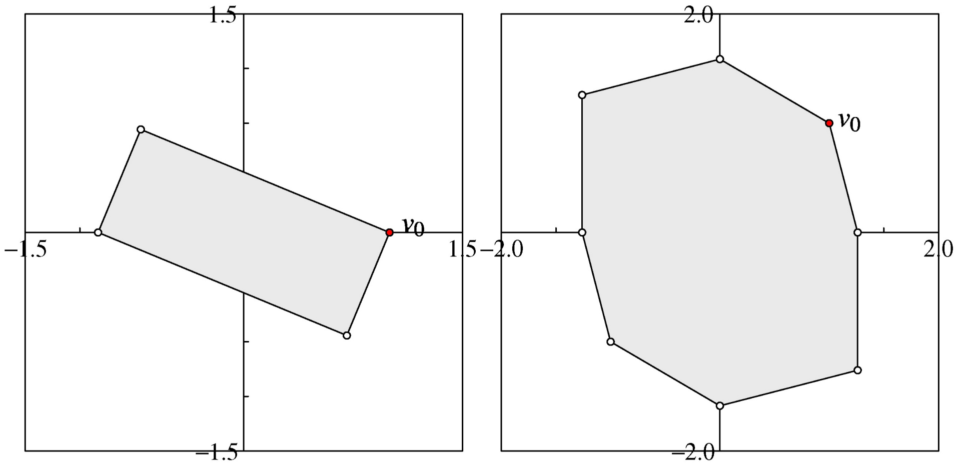


Figure A.10: Polytope norm for the pairs $\{A = F_1, B = F_5\}$ (left) and $\{A = F3, B = F5\}$ (right).

- $A = F_3$, $B = F_j$ $(j = 2, 4)$.

  Since $\rho(A) = \rho(B) = \|A\|_2 = \|B\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that both A and B are s.m.p.'s.

- $A = F_3$, $B = F_5$.

  l^1 norm proves to be extremal, $\overline{\rho}_3(\mathscr{F}) = \rho(A^2B)^{1/3} = \widehat{\rho}_3(\mathscr{F}) = \|A^2B\|_1^{1/3}$. $P = A^2B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 4^{1/3}$.

- $A = F_5$, $B = F_2$.

  We have that l^1 norm is extremal, $\overline{\rho}_3(\mathscr{F}) = \rho(AB^2)^{1/3} = \widehat{\rho}_3(\mathscr{F}) = \|AB^2\|_1^{1/3}$. $P = AB^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 4^{1/3}$.

- $A = F_5$, $B = F_4$.

  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.

- $A = F_5$, $B = F_6$.

  Since $\|A\|_1 = \|B\|_1 = 2$ and $\rho(AB)^{1/2} = 2$, we have that $\rho(\mathscr{F}) = 2$ and that $P = AB$ is an s.m.p.

- $A = F_8$, $B \in \mathbf{F}$.

  Since $\rho(A) = \|A\|_1 = \|B\|_1 = 2$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p.

Appendix B

Numerical results for the consensus

First example

Let us consider 20 agents and the following 35 configurations $\mathscr{A}_i$ of their neighborhoods.

$$\mathscr{A}_1 = \left\{ \begin{array}{lllllll} N_1 & = & \{1,7,8,14,15,16,17,18,20\} & N_{11} & = & \{1,5,9,11,12,15,16\} \\ N_2 & = & \{2,3\} & N_{12} & = & \{2,5,6,8,12,16\} \\ N_3 & = & \{1,3,8,10,17,19\} & N_{13} & = & \{6,7,13,18,20\} \\ N_4 & = & \{4,11,13,14,17\} & N_{14} & = & \{2,3,4,9,11,13,14,16\} \\ N_5 & = & \{3,5,7,10,15,18,20\} & N_{15} & = & \{2,3,6,7,9,12,15,18\} \\ N_6 & = & \{1,6,9,10,11,17\} & N_{16} & = & \{1,5,7,9,11,12,15,16,18,20\} \\ N_7 & = & \{7,8,9\} & N_{17} & = & \{4,11,17,19\} \\ N_8 & = & \{4,8,9,13\} & N_{18} & = & \{11,17,18,19,20\} \\ N_9 & = & \{4,8,9,14,15,20\} & N_{19} & = & \{5,6,11,16,19\} \\ N_{10} & = & \{2,8,9,10,11,17\} & N_{20} & = & \{2,3,7,8,14,16,18,20\} \end{array} \right.$$

$$\mathscr{A}_2 = \left\{ \begin{array}{lllllll} N_1 & = & \{1,6,16,19\} & N_{11} & = & \{11,15,17\} \\ N_2 & = & \{2,3,5,10,17,20\} & N_{12} & = & \{4,8,11,12,15,16,17,19,20\} \\ N_3 & = & \{3,4,14,15,19,20\} & N_{13} & = & \{5,11,13,15,16\} \\ N_4 & = & \{4,6,12,20\} & N_{14} & = & \{11,14,16\} \\ N_5 & = & \{3,5,7,10,12,14,18\} & N_{15} & = & \{3,5,13,15,16\} \\ N_6 & = & \{6,8,15,16,18,19,20\} & N_{16} & = & \{1,9,12,13,16,19,20\} \\ N_7 & = & \{5,6,7,10,19\} & N_{17} & = & \{7,11,17\} \\ N_8 & = & \{2,4,6,8,15,20\} & N_{18} & = & \{7,10,15,18,20\} \\ N_9 & = & \{2,5,8,9,13,15,17,20\} & N_{19} & = & \{1,2,7,11,16,18,19\} \\ N_{10} & = & \{1,8,10,11,15,16,18\} & N_{20} & = & \{1,3,4,7,9,11,13,17,18,20\} \end{array} \right.$$

$$\mathscr{A}_3 = \begin{cases} N_1 = \{1,6,8,15,17,20\} & N_{11} = \{5,6,8,9,10,11,12,14,19\} \\ N_2 = \{1,2,4,8,9,13,16,17,20\} & N_{12} = \{1,3,4,6,7,9,12,15,16,20\} \\ N_3 = \{1,3,5,8,12,13,15,17,19,20\} & N_{13} = \{1,5,7,8,13\} \\ N_4 = \{2,3,4,5,8,11,14,16,17\} & N_{14} = \{4,7,14,16,20\} \\ N_5 = \{3,5,7,9,15,17,20\} & N_{15} = \{11,15\} \\ N_6 = \{4,6,15\} & N_{16} = \{2,3,4,10,16,19\} \\ N_7 = \{1,7,8,11,16,19,20\} & N_{17} = \{2,6,13,16,17,18,19\} \\ N_8 = \{8,11,14,18,20\} & N_{18} = \{2,4,17,18,20\} \\ N_9 = \{2,4,6,9,12,14,16\} & N_{19} = \{6,7,9,12,19\} \\ N_{10} = \{2,6,10,11\} & N_{20} = \{4,5,6,10,19,20\} \end{cases}$$

$$\mathscr{A}_4 = \begin{cases} N_1 = \{1,18,20\} & N_{11} = \{7,11,12,16,17,20\} \\ N_2 = \{2,4,6,7,11\} & N_{12} = \{12\} \\ N_3 = \{3,7,17,18,20\} & N_{13} = \{5,9,12,13,19,20\} \\ N_4 = \{4,8\} & N_{14} = \{11,14,18\} \\ N_5 = \{5,9,15,17\} & N_{15} = \{1,3,12,14,15,16,20\} \\ N_6 = \{5,6\} & N_{16} = \{4,12,15,16,20\} \\ N_7 = \{1,3,4,7,12,13,14\} & N_{17} = \{3,4,13,16,17,18,20\} \\ N_8 = \{5,6,8,13,17,20\} & N_{18} = \{2,8,15,18,20\} \\ N_9 = \{4,9,11,13,14,17,19\} & N_{19} = \{9,16,19,20\} \\ N_{10} = \{3,5,7,10,13,17,18\} & N_{20} = \{4,5,8,15,20\} \end{cases}$$

$$\mathscr{A}_5 = \begin{cases} N_1 = \{1,2,5,8,16,18,20\} & N_{11} = \{1,5,6,11,17,18\} \\ N_2 = \{1,2,5,15,16\} & N_{12} = \{3,7,12,15,20\} \\ N_3 = \{3,4,8,9,12,13,14,15\} & N_{13} = \{1,3,10,12,13,14,17\} \\ N_4 = \{2,4,7,9,11,12,19\} & N_{14} = \{1,3,14,15\} \\ N_5 = \{3,5,6,7,13\} & N_{15} = \{4,5,8,10,15\} \\ N_6 = \{1,2,6,8,20\} & N_{16} = \{6,11,12,16,17,18,20\} \\ N_7 = \{7,16,18\} & N_{17} = \{16,17,18,19\} \\ N_8 = \{8,10,11,18\} & N_{18} = \{4,8,10,16,18,19\} \\ N_9 = \{3,7,9,10,17,18,20\} & N_{19} = \{1,3,4,6,7,10,12,14,18,19\} \\ N_{10} = \{6,8,10,17,19\} & N_{20} = \{2,4,9,16,20\} \end{cases}$$

$$\mathscr{A}_6 = \begin{cases} N_1 = \{1,2,4,8,15,17\} & N_{11} = \{2,6,10,11,17,19\} \\ N_2 = \{2\} & N_{12} = \{9,10,12,16\} \\ N_3 = \{3,7,20\} & N_{13} = \{6,12,13,14,19\} \\ N_4 = \{3,4,7,16\} & N_{14} = \{4,5,8,12,14,16\} \\ N_5 = \{1,2,4,5,7,11,17,18\} & N_{15} = \{9,14,15\} \\ N_6 = \{6,9,14\} & N_{16} = \{1,4,5,11,12,15,16\} \\ N_7 = \{4,7,8,13\} & N_{17} = \{4,6,7,11,17,20\} \\ N_8 = \{2,8,12,15,19,20\} & N_{18} = \{3,6,12,13,14,18\} \\ N_9 = \{6,8,9,14,15,18,19\} & N_{19} = \{2,4,16,19,20\} \\ N_{10} = \{1,2,7,10,13,16,17,19,20\} & N_{20} = \{2,9,14,16,20\} \end{cases}$$

$$\mathscr{A}_7 = \left\{ \begin{array}{lllllll} N_1 & = & \{1,3,6,9,10,18,19\} & N_{11} & = & \{1,2,6,8,10,11,16,17,19\} \\ N_2 & = & \{2,5,7,8,11,17\} & N_{12} & = & \{12,15,20\} \\ N_3 & = & \{1,3,8,9,10,16,19,20\} & N_{13} & = & \{3,5,11,12,13,16,17\} \\ N_4 & = & \{4,5,12,17,20\} & N_{14} & = & \{1,2,4,7,10,11,12,14\} \\ N_5 & = & \{5,6,7,16\} & N_{15} & = & \{1,4,5,6,8,12,13,15\} \\ N_6 & = & \{1,6,8,10,11,12,14\} & N_{16} & = & \{1,3,10,15,16,17,20\} \\ N_7 & = & \{1,7,11,19\} & N_{17} & = & \{12,14,15,17,19,20\} \\ N_8 & = & \{1,5,8,15,19\} & N_{18} & = & \{1,16,18\} \\ N_9 & = & \{2,5,9,15,18,20\} & N_{19} & = & \{1,5,8,10,15,19\} \\ N_{10} & = & \{2,10,15\} & N_{20} & = & \{1,7,10,15,20\} \end{array} \right.$$

$$\mathscr{A}_8 = \left\{ \begin{array}{lllllll} N_1 & = & \{1,8,11,18\} & N_{11} & = & \{4,6,11,14,20\} \\ N_2 & = & \{2,12,16\} & N_{12} & = & \{3,7,11,12\} \\ N_3 & = & \{3,4,5,10,13\} & N_{13} & = & \{7,11,13,15,16,19,20\} \\ N_4 & = & \{4,5,6,7,10,12,14,16\} & N_{14} & = & \{9,10,11,14,18,19\} \\ N_5 & = & \{5,9,12\} & N_{15} & = & \{6,10,15,17,19\} \\ N_6 & = & \{3,6,9,13,16,17,18,19\} & N_{16} & = & \{1,4,12,13,15,16,20\} \\ N_7 & = & \{6,7,14,15\} & N_{17} & = & \{3,11,16,17,18\} \\ N_8 & = & \{1,6,8,12,17,18\} & N_{18} & = & \{1,5,18,20\} \\ N_9 & = & \{5,6,9,10,16,19\} & N_{19} & = & \{2,6,8,11,13,19\} \\ N_{10} & = & \{4,8,10,13,19\} & N_{20} & = & \{2,3,5,6,11,18,20\} \end{array} \right.$$

$$\mathscr{A}_9 = \left\{ \begin{array}{lllllll} N_1 & = & \{1,8,9,10,11\} & N_{11} & = & \{7,10,11,16,20\} \\ N_2 & = & \{2,6,7,8,13,14,18,20\} & N_{12} & = & \{4,6,12,18\} \\ N_3 & = & \{3,6,8,11,15,19\} & N_{13} & = & \{5,11,12,13\} \\ N_4 & = & \{4,7\} & N_{14} & = & \{1,3,5,14\} \\ N_5 & = & \{1,2,3,5\} & N_{15} & = & \{6,7,8,11,15\} \\ N_6 & = & \{6,13\} & N_{16} & = & \{2,3,4,6,8,11,16,17,19,20\} \\ N_7 & = & \{1,7,8\} & N_{17} & = & \{7,10,17,18\} \\ N_8 & = & \{2,8,9,13,19\} & N_{18} & = & \{5,8,13,16,18,19\} \\ N_9 & = & \{1,5,9,12,16,17\} & N_{19} & = & \{11,14,19\} \\ N_{10} & = & \{4,5,10,13,19\} & N_{20} & = & \{1,2,3,10,12,13,19,20\} \end{array} \right.$$

$$\mathscr{A}_{10} = \left\{ \begin{array}{lllllll} N_1 & = & \{1,2,4,5,7,14,20\} & N_{11} & = & \{2,4,6,11,18\} \\ N_2 & = & \{2,4,6\} & N_{12} & = & \{10,12,13,17,19\} \\ N_3 & = & \{1,2,3,6,12,13,17,18\} & N_{13} & = & \{6,11,13,19,20\} \\ N_4 & = & \{1,2,4,5,9,14,19\} & N_{14} & = & \{3,6,13,14,18,20\} \\ N_5 & = & \{1,3,5,9,13\} & N_{15} & = & \{14,15,17,19,20\} \\ N_6 & = & \{1,6,10,13\} & N_{16} & = & \{2,3,6,7,13,15,16,20\} \\ N_7 & = & \{4,7,11,20\} & N_{17} & = & \{5,7,10,11,15,16,17\} \\ N_8 & = & \{8,10,12,19\} & N_{18} & = & \{1,5,9,12,15,17,18\} \\ N_9 & = & \{6,7,9,10,12,15,17\} & N_{19} & = & \{1,5,8,13,19\} \\ N_{10} & = & \{3,5,10,20\} & N_{20} & = & \{2,4,11,13,14,15,17,20\} \end{array} \right.$$

$$\mathscr{A}_{11} = \left\{ \begin{array}{lcl lcl} N_1 & = & \{1,4,13\} & N_{11} & = & \{5,6,11,13,17\} \\ N_2 & = & \{2,8,11,15\} & N_{12} & = & \{3,5,9,12\} \\ N_3 & = & \{2,3,4,5,7,9,13,14,17\} & N_{13} & = & \{1,5,9,13\} \\ N_4 & = & \{3,4,6,8,9,20\} & N_{14} & = & \{1,4,11,14,15\} \\ N_5 & = & \{3,4,5,11,17,19\} & N_{15} & = & \{5,6,8,10,12,15,17,18\} \\ N_6 & = & \{5,6,14,19\} & N_{16} & = & \{4,9,10,13,14,15,16\} \\ N_7 & = & \{3,6,7,8,9,11,12,20\} & N_{17} & = & \{2,3,5,7,10,11,12,16,17,19\} \\ N_8 & = & \{6,8,18\} & N_{18} & = & \{4,7,15,16,17,18\} \\ N_9 & = & \{5,9,11,13,14,16\} & N_{19} & = & \{2,3,9,10,12,15,16,19\} \\ N_{10} & = & \{5,6,8,10,11,12,15,19\} & N_{20} & = & \{8,9,12,14,15,17,18,19,20\} \end{array} \right.$$

$$\mathscr{A}_{12} = \left\{ \begin{array}{lcl lcl} N_1 & = & \{1,3,9,14,15,16,19\} & N_{11} & = & \{3,6,11,13,19\} \\ N_2 & = & \{1,2,8,9,11,12,15,20\} & N_{12} & = & \{3,6,7,8,10,12,18\} \\ N_3 & = & \{3,12,20\} & N_{13} & = & \{2,11,13,15,17\} \\ N_4 & = & \{4,5,6,9,10,12,17,20\} & N_{14} & = & \{2,9,10,11,14,16\} \\ N_5 & = & \{5,14\} & N_{15} & = & \{5,11,15,18,19\} \\ N_6 & = & \{3,6,7,8,12\} & N_{16} & = & \{1,4,5,12,13,16,17\} \\ N_7 & = & \{2,7,9,12,18\} & N_{17} & = & \{1,3,4,10,11,13,15,17\} \\ N_8 & = & \{1,3,5,8,12,17,19\} & N_{18} & = & \{3,4,12,15,18\} \\ N_9 & = & \{2,7,9,12,17,19,20\} & N_{19} & = & \{4,6,7,12,19\} \\ N_{10} & = & \{6,7,8,10,12,13,15\} & N_{20} & = & \{7,9,18,20\} \end{array} \right.$$

$$\mathscr{A}_{13} = \left\{ \begin{array}{lcl lcl} N_1 & = & \{1,6,9,12,20\} & N_{11} & = & \{2,3,11,16,20\} \\ N_2 & = & \{2,4,13,15,16\} & N_{12} & = & \{4,12,17,20\} \\ N_3 & = & \{3,4,5,14,15\} & N_{13} & = & \{2,9,10,13,16,17,19\} \\ N_4 & = & \{2,4,6,15,17,19\} & N_{14} & = & \{6,8,14,20\} \\ N_5 & = & \{5,7,12,13,15\} & N_{15} & = & \{8,10,13,14,15,17,19\} \\ N_6 & = & \{6,10,13,14,15,19\} & N_{16} & = & \{2,3,9,16,19,20\} \\ N_7 & = & \{2,7,11,12,16\} & N_{17} & = & \{3,6,7,8,9,12,17,18,19\} \\ N_8 & = & \{2,5,8,20\} & N_{18} & = & \{8,15,18\} \\ N_9 & = & \{3,4,5,9,10,12,14,17,18,19\} & N_{19} & = & \{3,5,7,11,12,15,18,19\} \\ N_{10} & = & \{1,10,16,19,20\} & N_{20} & = & \{1,6,20\} \end{array} \right.$$

$$\mathscr{A}_{14} = \left\{ \begin{array}{lcl lcl} N_1 & = & \{1,3,4,5,6,7,10,11,19\} & N_{11} & = & \{4,5,11,13,15,16,20\} \\ N_2 & = & \{2,7,10,14,18,20\} & N_{12} & = & \{4,8,12,13,16,19,20\} \\ N_3 & = & \{3,6,7,11,15\} & N_{13} & = & \{3,4,9,13,14,17,18\} \\ N_4 & = & \{4,7,10,15,18,19\} & N_{14} & = & \{1,14,17\} \\ N_5 & = & \{1,2,5,9,10,14\} & N_{15} & = & \{5,10,11,15,18,19\} \\ N_6 & = & \{1,2,3,6\} & N_{16} & = & \{6,16,18\} \\ N_7 & = & \{4,7\} & N_{17} & = & \{6,8,10,12,14,15,17,18,20\} \\ N_8 & = & \{2,6,8,14,17,19\} & N_{18} & = & \{1,2,5,9,10,18\} \\ N_9 & = & \{1,5,9,12\} & N_{19} & = & \{1,9,10,12,19\} \\ N_{10} & = & \{1,2,10\} & N_{20} & = & \{7,9,12,13,20\} \end{array} \right.$$

$$\mathscr{A}_{15} = \begin{cases} N_1 = \{1,4,5,12,20\} & N_{11} = \{5,8,11,18\} \\ N_2 = \{2\} & N_{12} = \{5,6,7,12,13,14,16\} \\ N_3 = \{3,8,10,13,17,18,19\} & N_{13} = \{2,3,7,13,15,18\} \\ N_4 = \{2,4,5,8,11,16,19\} & N_{14} = \{10,14\} \\ N_5 = \{2,3,5,8,9,12,16,17,18,19\} & N_{15} = \{1,2,7,9,11,15,16,18,19,20\} \\ N_6 = \{1,2,3,4,6,8,11,12\} & N_{16} = \{1,4,11,15,16,17\} \\ N_7 = \{1,3,7,8,9,10,11,14,15,20\} & N_{17} = \{9,14,15,17,20\} \\ N_8 = \{8,9,13\} & N_{18} = \{1,8,14,16,17,18\} \\ N_9 = \{1,7,8,9,12,15,16\} & N_{19} = \{3,8,13,15,19,20\} \\ N_{10} = \{2,4,6,8,9,10,11,13\} & N_{20} = \{3,9,11,17,20\} \end{cases}$$

$$\mathscr{A}_{16} = \begin{cases} N_1 = \{1,6,9,10,17,20\} & N_{11} = \{5,11,16,20\} \\ N_2 = \{2,3,6,8,11,18\} & N_{12} = \{1,10,12,16\} \\ N_3 = \{2,3,5,7,14,15,19\} & N_{13} = \{4,5,8,10,11,13,16,17,19\} \\ N_4 = \{4,5,11,13,15,19\} & N_{14} = \{1,10,13,14,18,19\} \\ N_5 = \{2,5,6,16,18\} & N_{15} = \{10,12,15,17,18\} \\ N_6 = \{6,9,10,12,20\} & N_{16} = \{10,16\} \\ N_7 = \{1,4,7,8,9,19,20\} & N_{17} = \{1,5,6,7,9,10,14,17,18,20\} \\ N_8 = \{8,10,14\} & N_{18} = \{2,3,7,8,11,12,17,18,19\} \\ N_9 = \{9,13,15\} & N_{19} = \{1,5,16,19\} \\ N_{10} = \{3,5,9,10,12,15,16,18,19\} & N_{20} = \{1,3,8,9,12,16,17,19,20\} \end{cases}$$

$$\mathscr{A}_{17} = \begin{cases} N_1 = \{1,5,13,17\} & N_{11} = \{1,7,11,13,17\} \\ N_2 = \{1,2,3,4,10,13,15\} & N_{12} = \{5,12,15\} \\ N_3 = \{2,3,7,9,18\} & N_{13} = \{2,4,13,15,17,18\} \\ N_4 = \{2,4,5,9,10,12\} & N_{14} = \{5,7,8,9,14,15,18\} \\ N_5 = \{2,5,12,14,20\} & N_{15} = \{13,14,15,16\} \\ N_6 = \{6,10,13,17\} & N_{16} = \{7,9,12,13,16\} \\ N_7 = \{2,4,5,6,7,10,13,17\} & N_{17} = \{8,9,13,16,17,18,20\} \\ N_8 = \{8,9,14,17\} & N_{18} = \{3,5,6,13,17,18,20\} \\ N_9 = \{3,7,8,9,11,14,17,20\} & N_{19} = \{6,18,19\} \\ N_{10} = \{2,3,10,15\} & N_{20} = \{3,4,8,10,11,18,19,20\} \end{cases}$$

$$\mathscr{A}_{18} = \begin{cases} N_1 = \{1,2,8,10,11,15\} & N_{11} = \{8,10,11,12,19,20\} \\ N_2 = \{2,9,18,19\} & N_{12} = \{4,6,12,13,17,20\} \\ N_3 = \{2,3,10,11,15\} & N_{13} = \{3,5,6,13,14\} \\ N_4 = \{3,4,9,14,15\} & N_{14} = \{1,2,3,8,14,15,17\} \\ N_5 = \{1,5,6,18,19\} & N_{15} = \{2,5,6,8,11,15\} \\ N_6 = \{2,4,6,10,14,16\} & N_{16} = \{2,12,16,18,19\} \\ N_7 = \{2,7,17,18\} & N_{17} = \{1,3,17,18,20\} \\ N_8 = \{8,10,15,18\} & N_{18} = \{3,4,13,18,19,20\} \\ N_9 = \{1,5,9,12,16\} & N_{19} = \{1,2,3,11,15,19\} \\ N_{10} = \{5,6,10,11,16,19\} & N_{20} = \{1,8,10,12,17,20\} \end{cases}$$

$$\mathscr{A}_{19} = \left\{ \begin{array}{lclllcl} N_1 & = & \{1,4,13,14,18\} & & N_{11} & = & \{1,3,5,11,17\} \\ N_2 & = & \{2,7,19\} & & N_{12} & = & \{12,13,14,15,19\} \\ N_3 & = & \{2,3,18,19\} & & N_{13} & = & \{1,6,7,9,13,18,19\} \\ N_4 & = & \{1,4,7,8,15,16,18,20\} & & N_{14} & = & \{6,8,12,14\} \\ N_5 & = & \{5,6,8,15,17,18,20\} & & N_{15} & = & \{4,6,8,14,15,17,19\} \\ N_6 & = & \{5,6,7,9,10,16\} & & N_{16} & = & \{12,13,16,20\} \\ N_7 & = & \{7,10,16\} & & N_{17} & = & \{2,4,6,9,13,14,17\} \\ N_8 & = & \{5,7,8,10,11,12,14\} & & N_{18} & = & \{3,7,10,11,13,15,17,18,20\} \\ N_9 & = & \{4,9,17,18\} & & N_{19} & = & \{1,4,6,7,10,13,15,16,19\} \\ N_{10} & = & \{2,4,10,11,14,18\} & & N_{20} & = & \{1,8,9,10,11,13,20\} \end{array} \right.$$

$$\mathscr{A}_{20} = \left\{ \begin{array}{lclllcl} N_1 & = & \{1,2,3,10,15,18\} & & N_{11} & = & \{1,2,3,8,9,11,17,18,19,20\} \\ N_2 & = & \{2,3,4,5,6,7,8\} & & N_{12} & = & \{4,5,7,8,10,12,13,14\} \\ N_3 & = & \{3,6,12,13\} & & N_{13} & = & \{2,4,5,9,12,13\} \\ N_4 & = & \{4,13,18\} & & N_{14} & = & \{5,14,16,17\} \\ N_5 & = & \{5,8,14,16\} & & N_{15} & = & \{4,6,9,10,13,14,15,17\} \\ N_6 & = & \{3,6,8,9,14,16,19,20\} & & N_{16} & = & \{2,4,8,15,16,19\} \\ N_7 & = & \{6,7,8,9,15\} & & N_{17} & = & \{3,4,8,17\} \\ N_8 & = & \{4,8,17\} & & N_{18} & = & \{4,5,15,18,19,20\} \\ N_9 & = & \{7,8,9,10,12,14,15,19\} & & N_{19} & = & \{5,13,15,19,20\} \\ N_{10} & = & \{1,10,11,14,19\} & & N_{20} & = & \{3,11,12,17,20\} \end{array} \right.$$

$$\mathscr{A}_{21} = \left\{ \begin{array}{lclllcl} N_1 & = & \{1,2,4,16,18,20\} & & N_{11} & = & \{4,8,11,15,16,19\} \\ N_2 & = & \{2,6,9,14,18,19\} & & N_{12} & = & \{5,12,15,19\} \\ N_3 & = & \{3,6,7,11,18\} & & N_{13} & = & \{1,2,5,7,13,18\} \\ N_4 & = & \{1,4,9,11,12,13,17\} & & N_{14} & = & \{1,4,6,11,12,13,14,15,17\} \\ N_5 & = & \{5,9\} & & N_{15} & = & \{2,4,6,15,16,20\} \\ N_6 & = & \{6,8,16,20\} & & N_{16} & = & \{1,2,12,16\} \\ N_7 & = & \{6,7,10,19\} & & N_{17} & = & \{3,7,10,11,17,20\} \\ N_8 & = & \{2,6,8,12,16,18,20\} & & N_{18} & = & \{3,5,11,18\} \\ N_9 & = & \{7,9,16,20\} & & N_{19} & = & \{2,3,9,10,17,19,20\} \\ N_{10} & = & \{1,8,10,15,19,20\} & & N_{20} & = & \{1,2,4,5,7,8,9,20\} \end{array} \right.$$

$$\mathscr{A}_{22} = \left\{ \begin{array}{lclllcl} N_1 & = & \{1,8,13,19\} & & N_{11} & = & \{1,3,4,5,6,10,11,18\} \\ N_2 & = & \{1,2,4,12,15,20\} & & N_{12} & = & \{1,9,10,12\} \\ N_3 & = & \{1,2,3,5,7,17,19\} & & N_{13} & = & \{2,7,10,13,16,18,20\} \\ N_4 & = & \{1,2,4,18\} & & N_{14} & = & \{2,5,10,12,13,14\} \\ N_5 & = & \{2,5,9,14,15,16,19\} & & N_{15} & = & \{2,5,7,8,9,13,15,18\} \\ N_6 & = & \{1,6,10,12,15,16,19\} & & N_{16} & = & \{6,12,16,20\} \\ N_7 & = & \{2,7,8,13,15,16,18,19\} & & N_{17} & = & \{1,6,8,9,12,13,14,17\} \\ N_8 & = & \{2,8\} & & N_{18} & = & \{7,9,12,16,18,19\} \\ N_9 & = & \{3,6,8,9,11,16,17\} & & N_{19} & = & \{11,14,16,17,18,19\} \\ N_{10} & = & \{3,6,7,9,10,12,16,17\} & & N_{20} & = & \{1,6,8,14,17,18,20\} \end{array} \right.$$

$$\mathscr{A}_{23} = \begin{cases} N_1 = \{1,10,11,15,16,19\} & N_{11} = \{2,3,4,7,9,10,11\} \\ N_2 = \{2,4,8,11,15,17,18\} & N_{12} = \{1,3,4,7,9,11,12,15\} \\ N_3 = \{3,15\} & N_{13} = \{3,7,13,15\} \\ N_4 = \{4,5,7,11,12,16\} & N_{14} = \{4,7,14,20\} \\ N_5 = \{2,5,7,9,12,13,15,19\} & N_{15} = \{3,10,12,14,15,17,18\} \\ N_6 = \{2,3,6,7,11,12,13,15,17\} & N_{16} = \{5,11,12,15,16,17\} \\ N_7 = \{2,7,8,13,15,17,18,19\} & N_{17} = \{6,9,13,17,19,20\} \\ N_8 = \{1,3,5,6,7,8,19\} & N_{18} = \{1,2,5,14,16,18,19,20\} \\ N_9 = \{1,2,9,11,15,17,20\} & N_{19} = \{1,2,4,7,11,12,15,17,19\} \\ N_{10} = \{2,4,6,8,10,12\} & N_{20} = \{3,6,10,19,20\} \end{cases}$$

$$\mathscr{A}_{24} = \begin{cases} N_1 = \{1,2,3,4,9,11\} & N_{11} = \{2,7,10,11,20\} \\ N_2 = \{2,6,9,10,11,19,20\} & N_{12} = \{6,12,15\} \\ N_3 = \{3,4,5,6,16,20\} & N_{13} = \{6,12,13,15,17,19\} \\ N_4 = \{4,7,10,13,18\} & N_{14} = \{9,10,13,14,19\} \\ N_5 = \{1,3,4,5,7,9,11,14,15,16\} & N_{15} = \{1,2,7,9,12,13,15\} \\ N_6 = \{4,6,12,15\} & N_{16} = \{2,4,5,12,16,19\} \\ N_7 = \{1,4,5,7,9,19,20\} & N_{17} = \{1,3,7,17,18,20\} \\ N_8 = \{2,8,9,12,14,19\} & N_{18} = \{1,2,3,5,7,15,18\} \\ N_9 = \{1,2,5,7,9,12,13,14,15,17,19,20\} & N_{19} = \{1,3,4,15,17,19\} \\ N_{10} = \{2,3,8,10,19,20\} & N_{20} = \{1,2,3,11,15,17,18,20\} \end{cases}$$

$$\mathscr{A}_{25} = \begin{cases} N_1 = \{1,4,6,10,20\} & N_{11} = \{3,5,11,17,18,19\} \\ N_2 = \{1,2,3,4,6,10,14\} & N_{12} = \{3,4,5,8,9,12,14,17\} \\ N_3 = \{1,3,14,17,19,20\} & N_{13} = \{2,3,6,8,13,20\} \\ N_4 = \{4,9,10,13\} & N_{14} = \{3,6,13,14,15,17,19\} \\ N_5 = \{1,3,5,10,18,19\} & N_{15} = \{1,7,15,19\} \\ N_6 = \{6,14,17,19\} & N_{16} = \{6,7,16\} \\ N_7 = \{3,7,11,12,13,16,18,19,20\} & N_{17} = \{3,6,10,15,17\} \\ N_8 = \{2,3,8,10,19\} & N_{18} = \{4,8,9,14,18\} \\ N_9 = \{6,7,8,9,12\} & N_{19} = \{4,5,6,19,20\} \\ N_{10} = \{8,10,11,12,13,14,16,18\} & N_{20} = \{1,7,10,15,20\} \end{cases}$$

$$\mathscr{A}_{26} = \begin{cases} N_1 = \{1,2,9,12,14,18\} & N_{11} = \{1,3,11,13,15,16,20\} \\ N_2 = \{2,7,17\} & N_{12} = \{5,6,12,15,18,20\} \\ N_3 = \{1,3,6,10,17\} & N_{13} = \{13,15\} \\ N_4 = \{2,3,4,5,6,17,20\} & N_{14} = \{1,4,5,6,7,14,17\} \\ N_5 = \{5,6,19\} & N_{15} = \{4,5,7,11,12,14,15\} \\ N_6 = \{3,5,6,9,10,11,18,20\} & N_{16} = \{4,5,11,16,17,20\} \\ N_7 = \{1,3,6,7,10,13,18\} & N_{17} = \{2,3,4,5,6,9,12,15,16,17\} \\ N_8 = \{3,7,8,9,14,15,16,17\} & N_{18} = \{3,17,18\} \\ N_9 = \{4,9,10,11,15\} & N_{19} = \{4,10,15,17,19,20\} \\ N_{10} = \{10,13,17,19,20\} & N_{20} = \{3,4,8,20\} \end{cases}$$

$$\mathscr{A}_{27} = \begin{cases} N_1 = \{1,2,4,5,8,12,14,15,20\} & N_{11} = \{1,4,7,8,9,11,14,20\} \\ N_2 = \{1,2,13,15,19\} & N_{12} = \{2,7,12,15\} \\ N_3 = \{2,3,7,12,13,16\} & N_{13} = \{1,2,4,5,13,14,15,16,19\} \\ N_4 = \{4,14,17,20\} & N_{14} = \{2,5,9,14\} \\ N_5 = \{5,10,12,13,19\} & N_{15} = \{5,10,14,15,16,19\} \\ N_6 = \{2,4,6,8,14,15,16\} & N_{16} = \{1,5,6,7,9,10,16\} \\ N_7 = \{2,3,5,7,10,11\} & N_{17} = \{5,9,12,13,15,17\} \\ N_8 = \{5,6,7,8\} & N_{18} = \{1,7,10,11,14,18\} \\ N_9 = \{1,4,7,9,10,17\} & N_{19} = \{1,5,14,19\} \\ N_{10} = \{6,8,10,13,18,20\} & N_{20} = \{11,20\} \end{cases}$$

$$\mathscr{A}_{28} = \begin{cases} N_1 = \{1,7,10,11\} & N_{11} = \{1,11,13,14,17\} \\ N_2 = \{2,5,6,14,16\} & N_{12} = \{2,6,12,14,17\} \\ N_3 = \{3,6,10\} & N_{13} = \{5,7,10,11,13,17,18,20\} \\ N_4 = \{2,4,6,12,18\} & N_{14} = \{3,7,14,16,19,20\} \\ N_5 = \{1,5,7,20\} & N_{15} = \{2,7,10,12,15\} \\ N_6 = \{1,2,6,20\} & N_{16} = \{3,7,10,12,16\} \\ N_7 = \{6,7,11,18\} & N_{17} = \{2,17,18\} \\ N_8 = \{3,4,6,7,8,10,19\} & N_{18} = \{3,5,9,12,16,18,19\} \\ N_9 = \{1,4,5,9,13\} & N_{19} = \{1,3,9,11,18,19\} \\ N_{10} = \{1,7,8,10,12\} & N_{20} = \{2,4,6,9,11,13,20\} \end{cases}$$

$$\mathscr{A}_{29} = \begin{cases} N_1 = \{1,9,12\} & N_{11} = \{11,17,18,20\} \\ N_2 = \{2,7,13,14,15,17,20\} & N_{12} = \{6,11,12,14,18\} \\ N_3 = \{2,3,11,14,15,16,19\} & N_{13} = \{2,8,10,12,13,17\} \\ N_4 = \{3,4,13,16,20\} & N_{14} = \{1,2,12,14,15,17\} \\ N_5 = \{5,7,8,12,13,15\} & N_{15} = \{11,12,15,17\} \\ N_6 = \{1,6,7,11,14,17,19\} & N_{16} = \{5,15,16\} \\ N_7 = \{5,7,9,12,15,16\} & N_{17} = \{1,3,9,11,15,17\} \\ N_8 = \{4,7,8,12\} & N_{18} = \{2,7,8,9,14,18,20\} \\ N_9 = \{9,11,16,20\} & N_{19} = \{5,12,13,18,19\} \\ N_{10} = \{1,2,6,10,13,14,16\} & N_{20} = \{2,3,4,5,6,7,17,18,20\} \end{cases}$$

$$\mathscr{A}_{30} = \begin{cases} N_1 = \{1,5,8,11,13,15,18,19\} & N_{11} = \{11,17\} \\ N_2 = \{2,8,9,14,18,19\} & N_{12} = \{2,5,7,8,11,12,14\} \\ N_3 = \{3,7,13,14\} & N_{13} = \{6,12,13\} \\ N_4 = \{4\} & N_{14} = \{1,6,14,16\} \\ N_5 = \{5,7,15,18\} & N_{15} = \{2,4,5,6,10,15,17,18\} \\ N_6 = \{2,6,10,11\} & N_{16} = \{1,2,16,17\} \\ N_7 = \{4,7,8,9,11,12,17\} & N_{17} = \{5,11,14,15,17\} \\ N_8 = \{3,8,17,18\} & N_{18} = \{2,7,13,14,15,17,18,20\} \\ N_9 = \{9,10,12,15,16,20\} & N_{19} = \{5,8,12,13,16,19\} \\ N_{10} = \{5,9,10,13,14\} & N_{20} = \{2,5,12,19,20\} \end{cases}$$

$$\mathscr{A}_{31} = \begin{cases} N_1 = \{1,6,9,14,15,16,19,20\} & N_{11} = \{3,7,11,13,14,15,16,20\} \\ N_2 = \{2,3,5,9,12,15,17\} & N_{12} = \{8,10,12,15,18\} \\ N_3 = \{1,2,3,7,9,11,12,20\} & N_{13} = \{4,6,8,10,13,14,16\} \\ N_4 = \{4,8,10,14,20\} & N_{14} = \{8,10,14\} \\ N_5 = \{2,4,5,9,20\} & N_{15} = \{1,4,5,15,19\} \\ N_6 = \{6,15,16,20\} & N_{16} = \{1,9,16\} \\ N_7 = \{3,6,7,8,9,14,16,17,19,20\} & N_{17} = \{8,11,12,15,17,19,20\} \\ N_8 = \{5,8,14,16,18,19,20\} & N_{18} = \{2,18\} \\ N_9 = \{1,8,9,13,19\} & N_{19} = \{2,7,10,11,19\} \\ N_{10} = \{9,10,13,17\} & N_{20} = \{3,11,15,17,19,20\} \end{cases}$$

$$\mathscr{A}_{32} = \begin{cases} N_1 = \{1,4,5,10\} & N_{11} = \{3,9,11,16,20\} \\ N_2 = \{2,11,13,15\} & N_{12} = \{2,3,6,11,12,13,17,18,19\} \\ N_3 = \{1,3,4,11,12,18\} & N_{13} = \{2,5,6,9,13,14,18,19\} \\ N_4 = \{2,3,4,6,8,10,13,14\} & N_{14} = \{6,9,14,16,17\} \\ N_5 = \{1,5,14,17,18\} & N_{15} = \{1,4,5,6,8,10,12,15,20\} \\ N_6 = \{3,6,9,12,16\} & N_{16} = \{2,5,7,12,14,15,16\} \\ N_7 = \{3,7,9,11,16,19\} & N_{17} = \{2,9,14,15,17,20\} \\ N_8 = \{3,4,8,16,20\} & N_{18} = \{2,3,7,8,16,18,20\} \\ N_9 = \{3,4,8,9,11,14,17\} & N_{19} = \{5,10,16,19,20\} \\ N_{10} = \{2,3,4,5,6,7,9,10,11,12,16\} & N_{20} = \{4,5,7,8,13,17,18,20\} \end{cases}$$

$$\mathscr{A}_{33} = \begin{cases} N_1 = \{1,4,11,20\} & N_{11} = \{3,4,6,7,11,13,15\} \\ N_2 = \{2,10,18\} & N_{12} = \{1,2,5,11,12,13,15,16,19,20\} \\ N_3 = \{2,3,9,19\} & N_{13} = \{2,7,8,12,13,18,20\} \\ N_4 = \{2,4,5,7,9,10,12,20\} & N_{14} = \{3,4,6,10,14,16,17,19,20\} \\ N_5 = \{2,5,6,14,15,16\} & N_{15} = \{4,8,12,15,17\} \\ N_6 = \{5,6,14,16\} & N_{16} = \{1,4,8,16,20\} \\ N_7 = \{5,7,8,17\} & N_{17} = \{2,3,4,7,10,15,17,18,19\} \\ N_8 = \{3,8,9,11,12,15,18\} & N_{18} = \{1,9,11,13,18\} \\ N_9 = \{6,9\} & N_{19} = \{1,2,11,12,16,18,19\} \\ N_{10} = \{9,10,12,19\} & N_{20} = \{4,5,8,10,13,14,16,18,20\} \end{cases}$$

$$\mathscr{A}_{34} = \begin{cases} N_1 = \{1,4,6\} & N_{11} = \{1,5,11,13,15,16,17\} \\ N_2 = \{2,7,8,11\} & N_{12} = \{2,5,6,8,9,12\} \\ N_3 = \{3,19\} & N_{13} = \{4,5,7,8,10,13,15,19\} \\ N_4 = \{1,4,8,10,11,20\} & N_{14} = \{1,5,9,10,13,14,16,17,18\} \\ N_5 = \{1,5,9,12,16,18,19\} & N_{15} = \{2,3,14,15,20\} \\ N_6 = \{2,6,7,10,16,18\} & N_{16} = \{2,9,11,12,13,16,17\} \\ N_7 = \{4,7,10,13,16,17\} & N_{17} = \{1,4,6,8,10,15,16,17\} \\ N_8 = \{7,8,9,15,18\} & N_{18} = \{9,12,15,18\} \\ N_9 = \{5,9,15,16,20\} & N_{19} = \{2,16,18,19\} \\ N_{10} = \{2,4,5,10,13\} & N_{20} = \{2,9,17,20\} \end{cases}$$

$$\mathscr{A}_{35} = \begin{cases} N_1 = \{1,6,9,18\} & N_{11} = \{2,9,11,12,17,19\} \\ N_2 = \{2,7,10,12,14,16\} & N_{12} = \{6,10,12,14,16\} \\ N_3 = \{2,3,4,5,19\} & N_{13} = \{8,10,11,12,13,16,18,19\} \\ N_4 = \{1,2,4,7,17,18\} & N_{14} = \{6,7,8,10,14\} \\ N_5 = \{1,5,6,13,15,16\} & N_{15} = \{1,2,3,5,8,11,12,15,17\} \\ N_6 = \{3,4,5,6,7,15,16,18\} & N_{16} = \{8,10,11,16,17,18\} \\ N_7 = \{2,4,7,10\} & N_{17} = \{2,11,15,17\} \\ N_8 = \{1,3,6,8,12\} & N_{18} = \{1,4,6,7,11,13,14,18\} \\ N_9 = \{4,6,7,9,13\} & N_{19} = \{1,12,13,19\} \\ N_{10} = \{8,10,12,18\} & N_{20} = \{4,5,6,9,10,11,17,18,20\} \end{cases}$$

The evolution matrix F_j, associated with $\mathscr{A}_j$ for $j = 1,\ldots,35$, is equal to the adjacency matrix A_j, corresponding to $\mathscr{A}_j$, premultiplied by the diagonal matrix $D_j = [d_{ii}]$, with d_{ii} cardinality of N_i (ref equation (3.11)). We define the *density* of a neighborhood N_i as the cardinality of N_i, not counting the agent i, divided by 19, which is the total number of possible neighbors of i. The set of evolution matrices $\mathscr{F} = \{F_j\}_{j=1}^{35}$ has a mean density around 0.249.

We observe that the matrices in the set with maximum spectral radius are those with a corresponding topology of the network such that one agent is the *leader* among all the agents, i.e. it has no outgoing links to the other agents, but there is a directed path from every agent in the network to it. This is the case of F_i with $i = \{4,6,15,30\}$. Their second spectral radii are

$$\begin{aligned} \rho_2(F_6) &= 0.953639710720261 \\ \rho_2(F_{15}) &= 0.964986919188626 \\ \rho_2(F_4) &= 0.965467413828792 \\ \rho_2(F_{30}) &= 0.984891122708641 \end{aligned}$$

If we make use of the Gripenberg algorithm presented in [74], which is based on the four members inequality (4.23), to study the bounded family $\mathscr{F}$, after 20 steps and choosing as operator norm the spectral norm, we obtain the following bounds for the second joint spectral radius of the set

$$\overline{\rho}_{10}(\mathscr{F}) = 0.984891 \leq \rho_2(\mathscr{F}) \leq 0.997385 = \widehat{\rho}_{20}(\mathscr{F}).$$

These bounds allow us to conclude that the family $\mathscr{F}$ has a second spectral radius strictly less then 1. Moreover, we observe that the lower bound is given by the second spectral radius of F_{30} which is, therefore, a candidate s.m.p.

Now, making use of the algorithm proposed by Protasov et al. in [57], we obtain an approximated extremal ellipsoidal norm for the family $\mathscr{F}$ which gives a tighter upper bound for $\rho_2(\mathscr{F})$.

We start solving for $k=2$ the optimization problem

$$\begin{aligned} &\min\ r \\ &\text{s.t.}\ \ X \ \succ\ 0, \\ &A^T X A \ \preceq\ rX, \quad \text{for all} \quad A \in \mathscr{P}_k(\mathscr{F}) \end{aligned} \tag{B.1}$$

where $\mathscr{P}_k(\mathscr{F})$ is the set of all the possible products of length k whose factors are elements of $\mathscr{F}$.

Applying a conic programming method together with the bisection method, we obtain the matrix X given by

$$X = \operatorname{diag}(w) - H \cdot 10^{-3}$$

where

$$H \approx \begin{pmatrix}
0 & 2.47 & 5.45 & 2.32 & 3.97 & 3.00 & 2.76 & 3.34 & 2.36 & 3.39 & 10.61 & 4.84 & 1.95 & 2.92 & 2.85 & 2.17 & 1.15 & 3.85 & 1.81 \\
2.47 & 0 & 4.44 & 2.17 & 2.98 & 1.73 & 2.82 & 2.25 & 1.90 & 2.61 & 2.53 & 2.77 & 1.49 & 2.83 & 1.82 & 1.19 & 2.29 & 2.83 & 2.88 \\
5.45 & 4.44 & 0 & 4.41 & -3.51 & 6.75 & 2.74 & 5.23 & 3.04 & 2.10 & 1.70 & -4.13 & 7.46 & 3.74 & 5.97 & 10.51 & 3.31 & 0.71 & 4.16 \\
2.32 & 2.17 & 4.41 & 0 & 3.10 & 2.20 & 2.29 & 2.51 & 2.13 & 3.29 & 7.09 & 3.21 & 1.69 & 2.53 & 2.43 & 1.48 & 1.41 & 2.98 & 1.79 \\
3.97 & 2.98 & -3.51 & 3.10 & 0 & 3.83 & 3.35 & 3.59 & 2.97 & 3.35 & 3.71 & 1.44 & 3.32 & 3.70 & 3.35 & 4.32 & 2.79 & 2.76 & 3.63 \\
3.00 & 1.73 & 6.75 & 2.20 & 3.83 & 0 & 3.23 & 2.56 & 2.70 & 3.96 & 6.94 & 4.40 & 1.08 & 3.61 & 2.38 & 1.28 & 2.13 & 3.28 & 2.90 \\
2.76 & 2.82 & 2.74 & 2.29 & 3.35 & 3.23 & 0 & 3.49 & 3.01 & 3.88 & 10.11 & 3.11 & 2.46 & 2.39 & 2.87 & 3.12 & 0.66 & 3.53 & 1.10 \\
3.34 & 2.25 & 5.23 & 2.51 & 3.59 & 2.56 & 3.49 & 0 & 2.81 & 3.72 & 3.87 & 3.36 & 1.61 & 3.63 & 2.25 & 1.75 & 2.64 & 3.32 & 3.44 \\
2.36 & 1.90 & 3.04 & 2.13 & 2.97 & 2.70 & 3.01 & 2.81 & 0 & 2.23 & 5.61 & 3.91 & 1.64 & 3.10 & 2.21 & 2.16 & 1.90 & 3.03 & 2.82 \\
3.39 & 2.61 & 2.10 & 3.29 & 3.35 & 3.96 & 3.88 & 3.72 & 2.23 & 0 & 9.43 & 5.22 & 2.67 & 4.59 & 4.08 & 2.84 & 2.99 & 3.73 & 4.02 \\
10.61 & 2.53 & 1.70 & 7.09 & 3.71 & 6.94 & 10.11 & 3.87 & 5.61 & 9.43 & 0 & 5.64 & 2.99 & 13.06 & 7.29 & 2.75 & 12.32 & 3.74 & 14.69 \\
4.84 & 2.77 & -4.13 & 3.21 & 1.44 & 4.40 & 3.11 & 3.36 & 3.91 & 5.22 & 5.64 & 0 & 3.28 & 4.05 & 4.07 & 4.37 & 2.81 & 2.31 & 3.36 \\
1.95 & 1.49 & 7.46 & 1.69 & 3.32 & 1.08 & 2.46 & 1.61 & 1.64 & 2.67 & 2.99 & 3.28 & 0 & 2.51 & 1.34 & -0.06 & 1.93 & 2.67 & 2.38 \\
2.92 & 2.83 & 3.74 & 2.53 & 3.70 & 3.61 & 2.39 & 3.63 & 3.10 & 4.59 & 13.06 & 4.05 & 2.51 & 0 & 3.57 & 2.91 & 0.74 & 3.78 & 1.14 \\
2.85 & 1.82 & 5.97 & 2.43 & 3.35 & 2.38 & 2.87 & 2.25 & 2.21 & 4.08 & 7.29 & 4.07 & 1.34 & 3.57 & 0 & 0.71 & 2.41 & 2.96 & 2.88 \\
2.17 & 1.19 & 10.51 & 1.48 & 4.32 & 1.28 & 3.12 & 1.75 & 2.16 & 2.84 & 2.75 & 4.37 & -0.06 & 2.91 & 0.71 & 0 & 1.96 & 3.18 & 2.78 \\
1.15 & 2.29 & 3.31 & 1.41 & 2.79 & 2.13 & 0.66 & 2.64 & 1.90 & 2.99 & 12.32 & 2.81 & 1.93 & 0.74 & 2.41 & 1.96 & 0 & 2.86 & -1.01 \\
3.85 & 2.83 & 0.71 & 2.98 & 2.76 & 3.28 & 3.53 & 3.32 & 3.03 & 3.73 & 3.74 & 2.31 & 2.67 & 3.78 & 2.96 & 3.18 & 2.86 & 0 & 3.66 \\
1.81 & 2.88 & 4.16 & 1.79 & 3.63 & 2.90 & 1.10 & 3.44 & 2.82 & 4.02 & 14.69 & 3.36 & 2.38 & 1.14 & 2.88 & 2.78 & -1.01 & 3.66 & 0
\end{pmatrix}$$

$$w \approx \begin{pmatrix} 1.5551 & 1.5583 & 1.8478 & 1.5579 & 1.5611 & 1.5518 & 1.5561 & 1.5515 & 1.5604 & 1.5440 & 1.6066 & 1.5719 & 1.5727 & 1.5508 & 1.5495 & 1.5803 & 1.5880 & 1.5473 & 1.5891 \end{pmatrix}$$

Associated to the symmetric and positive definite matrix X there is the ellipsoidal norm $\|\cdot\|_X$, defined in (4.5), which is the approximation of an extremal norm for the family $\mathscr{F}$.

Hence, if we run the Gripenberg algorithm choosing as operator norm the ellipsoidal norm $\|\cdot\|_X$, after 10 steps we obtain

$$\overline{\rho}_{10}(\mathscr{F}) \;=\; 0.984891 \;\leq\; \rho_2(\mathscr{F}) \;\leq\; 0.990199 \;=\; \widehat{\rho}_{10}(\mathscr{F})$$

and, after 20 steps,

$$\overline{\rho}_{20}(\mathscr{F}) \;=\; 0.984891 \;\leq\; \rho_2(\mathscr{F}) \;\leq\; 0.987545 \;=\; \widehat{\rho}_{20}(\mathscr{F}).$$

Let us consider now equation (3.12). If we choose as initial condition

$$\begin{aligned}\theta(0)^T = (&0.07280\ , 0.28590, 0.43005, 1.00000, 0.01876,\\ &0.05239, 0.20579, 0.79045, 0.89105, 0.78879,\\ &0.54911, 0.66611, 0.36084, 0.90684, 0.23010,\\ &0.83631, 0.22347, 0.44872, 0.76184, 0.95011)\end{aligned}$$

such that $\|\theta(0)\|_\infty = 1$ and with mean value equal to 0.523471, after t steps the solution of (3.12) is given by

$$\theta(t) = F_{i_{t-1}} \cdot \ldots \cdot F_{i_1} \cdot F_{i_0} \theta(0) \tag{B.2}$$

where the sequence $\bar{i} = (i_0, i_1, \ldots, i_{t-1})$ takes values in the set $\{1, \ldots, 35\}$.

Let us consider $\theta^{(1)}(t)$ solution of (B.2) corresponding to the previously mentioned $\theta(0)$ and to the sequence of indices given by $\bar{i}^{(1)} = (30, 30, \ldots)$. It results that $\theta_4^{(1)}(t) = 1$ for every $t \geq 0$ and the mean value of $\theta^{(1)}(t)$ is around 0.999733 for $t = 500$ and 0.999999868 for $t = 1000$: the solution is slowly converging to e.

On the contrary, if we consider a generic sequence like

$$\bar{i}^{(2)} = (12, 7, 33, 5, 30, 3, 19, 21, 14, 7, \ldots),$$

after 10 steps the system has almost reached a global consensus. In particular the mean value of the solution $\theta^{(2)}(10)$ is around 0.502774 and the gap between its minimum and maximum value is $5.7 \cdot 10^{-5}$.

For every generic product we have, in a similar way, convergence after a few steps. This is due to the fact that a generic product of matrices belonging to $\mathscr{F}$ presents a second spectral radius which is, usually, less than 0.5.

Second example

For the second example we consider 125 agents and the following 10 configurations $\mathscr{A}_i$ of their neighborhoods.

$$
\mathscr{A}_1 = \left\{ \begin{array}{lcl}
N_1 & = & \{1,11,22,77,83,87\} \\
N_2 & = & \{2,9,10,18,79,82,87,101,104,112,115\} \\
N_3 & = & \{3,7,21,28,33,38,39,42,53,54,73,98,108,110,120\} \\
N_4 & = & \{4,19,26,30,35,60,86,87,101,114\} \\
N_5 & = & \{3,5,6,10,16,66,75,81,84,113\} \\
N_6 & = & \{1,5,6,16,22,35,41,55,68,77,87,101\} \\
N_7 & = & \{7,9,18,30,56,60,64,79,84,86,104,123,124\} \\
N_8 & = & \{8,18,26,40,74,86,100\} \\
N_9 & = & \{7,9,10,15,18,22,42,70,73,79,88,89,93,103\} \\
N_{10} & = & \{1,10,29,39,40,48,49,55,71,80,91,95,121,122,125\} \\
N_{11} & = & \{4,8,11,17,21,24,28,30,34,48,49,54,63,70,94,102,118\} \\
N_{12} & = & \{8,12,14,18,22,41,81\} \\
N_{13} & = & \{1,13,16,34,42,47,59,80,111\} \\
N_{14} & = & \{14,35,36,57,63,74,96\} \\
N_{15} & = & \{3,15,23,33,34,37,42,73,74,81,91,97,110,122\} \\
N_{16} & = & \{4,16,26,77,82\} \\
N_{17} & = & \{7,17,28,37,42,64,78,109,123\} \\
N_{18} & = & \{12,14,15,18,23,32,34,64,88,104,113\} \\
N_{19} & = & \{4,18,19,21,22,23,24,44,64,66,72,74,76,77,87,93,102,113,114,125\} \\
N_{20} & = & \{20,39,53,56,57,70,84,120\} \\
N_{21} & = & \{21,32,43,50,68,90,91,104\} \\
N_{22} & = & \{4,16,22,28,39,42,53,58,64,71,88,123\} \\
N_{23} & = & \{3,10,23,38,67,85,90,102,112\} \\
N_{24} & = & \{16,19,22,24,25,27,30,33,34,49,68,77,79,84,92,99,103\} \\
N_{25} & = & \{10,14,17,22,24,25,32,41,61,71,76,86,112\} \\
N_{26} & = & \{4,26,38,50,53,61,70,73,100,110,111,117,125\} \\
N_{27} & = & \{2,8,27,37,48,60,72,79,93,112\} \\
N_{28} & = & \{9,16,28,29,45,60,68,77,79,82,100,110,119\} \\
N_{29} & = & \{2,29,45,67,78,83,100,101,102\} \\
N_{30} & = & \{15,27,30,51,61,94,101,108,115\} \\
N_{31} & = & \{1,6,10,31,43,79,80,106,108\} \\
N_{32} & = & \{15,16,19,27,31,32,42,78,79,84,97,116\} \\
N_{33} & = & \{30,32,33,37,74,84,103,114,116\} \\
N_{34} & = & \{31,34,43,45,61,66,84,89,92,93,113\} \\
N_{35} & = & \{1,5,35,39,41,45,106,108,112\} \\
N_{36} & = & \{19,29,30,36,47,75,77,90,94,101,107\} \\
N_{37} & = & \{19,22,23,37,50,70,87,93,103,113,121\} \\
N_{38} & = & \{1,11,31,38,54,76,90,107\} \\
N_{39} & = & \{4,12,13,15,26,39,71,87,119\} \\
N_{40} & = & \{9,33,39,40,52,61,79,85,109,114,117,118\} \\
N_{41} & = & \{1,2,9,20,30,31,34,41,47,54,56,58,59,67,85,107,116,124\} \\
N_{42} & = & \{10,14,20,42,52,64,67,72,88,99,103\} \\
N_{43} & = & \{43,45,48,56,69,92,100,103,107,113\} \\
N_{44} & = & \{8,26,28,44,60,68,108,125\} \\
N_{45} & = & \{5,9,13,19,33,35,36,45,46,74,80,92,95,101,105,106\} \\
N_{46} & = & \{1,6,12,13,24,41,46,77,85,89,97,111\} \\
N_{47} & = & \{4,14,18,21,33,47,64,85,90,99,101,111\} \\
N_{48} & = & \{11,16,42,48,51,68,75,88,106,112\} \\
N_{49} & = & \{20,39,49,67,86\} \\
N_{50} & = & \{8,35,41,48,50,54,62,70,93,97,105\} \\
N_{51} & = & \{9,17,36,39,51,56,59,61,71,87,92,104\} \\
N_{52} & = & \{9,15,23,33,52,90,93,95,99,111\} \\
N_{53} & = & \{26,31,32,53,59,62,94,95,111,119\} \\
N_{54} & = & \{10,18,30,42,54,66,125\} \\
N_{55} & = & \{1,8,15,17,35,55,64,85,97,110\} \\
N_{56} & = & \{5,8,10,23,25,27,41,56,66,70,87,95,107,119\} \\
N_{57} & = & \{4,18,23,30,31,35,57,60,69,87,107,108,119\} \\
N_{58} & = & \{3,30,35,58,59,63,71,96,119\} \\
N_{59} & = & \{1,20,28,56,59,62,87,99,102,113,118\} \\
N_{60} & = & \{2,7,34,60,61,80,86,112,122\} \\
N_{61} & = & \{19,37,42,46,61,64,68,69,101,106,111,117,120\} \\
N_{62} & = & \{3,32,50,62,69,83,96,114,117\} \\
N_{63} & = & \{3,14,20,23,42,46,63,78,79,91,101,107\} \\
N_{64} & = & \{14,19,27,64,85,101,125\} \\
N_{65} & = & \{24,40,65,89,97,99,112\} \\
N_{66} & = & \{6,8,13,33,35,46,59,66,74,85,94,97,98\} \\
N_{67} & = & \{67,104,118\} \\
N_{68} & = & \{5,7,9,14,32,44,51,57,68,100,107,117\} \\
N_{69} & = & \{2,3,19,69,109\} \\
N_{70} & = & \{26,31,32,33,46,56,58,65,70,72,74,107,119,124\} \\
N_{71} & = & \{3,4,13,27,32,35,36,45,71,73,87,92,105,107,108,125\} \\
N_{72} & = & \{25,28,42,44,45,57,71,72,86\} \\
N_{73} & = & \{36,41,73,105,112,113\} \\
N_{74} & = & \{8,13,19,74,123\} \\
N_{75} & = & \{4,21,26,75,91,118\} \\
N_{76} & = & \{3,9,13,21,25,29,73,76,79,80,96,107\} \\
N_{77} & = & \{27,31,50,67,69,77,81,84,97,106,109,119\} \\
N_{78} & = & \{17,22,38,42,45,73,78,101,109\} \\
N_{79} & = & \{26,27,59,73,79,87,93\} \\
N_{80} & = & \{25,29,37,38,43,45,46,65,68,73,74,76,78,80,82,110,118\} \\
N_{81} & = & \{1,20,56,60,66,80,81,90,107,108,115,122\} \\
N_{82} & = & \{8,29,40,75,76,82,85,93\} \\
N_{83} & = & \{9,14,42,46,63,83,85,107,109\} \\
N_{84} & = & \{5,14,35,83,84,100,122\} \\
N_{85} & = & \{9,26,44,48,62,64,67,85,110\} \\
N_{86} & = & \{23,37,61,62,81,86,106\} \\
N_{87} & = & \{14,81,87,95,98,114,119,124\} \\
N_{88} & = & \{1,10,27,51,65,67,70,88,91,99,107,121\} \\
N_{89} & = & \{7,8,26,37,49,89,109,116,122,124\} \\
N_{90} & = & \{35,47,49,55,56,81,90,97\} \\
N_{91} & = & \{16,40,61,64,91,116,122\} \\
N_{92} & = & \{11,15,21,23,57,60,63,66,77,92,101,107,123\} \\
N_{93} & = & \{23,30,52,64,93,102,106,119\} \\
N_{94} & = & \{10,21,22,35,39,61,94\} \\
N_{95} & = & \{5,32,39,63,64,75,76,77,95,112,125\} \\
N_{96} & = & \{11,31,35,42,44,66,96,106,113,117\} \\
N_{97} & = & \{7,12,33,38,64,89,94,96,97,99,124\} \\
N_{98} & = & \{2,3,24,77,80,87,98,99,111,123\} \\
N_{99} & = & \{3,13,15,29,35,46,73,83,89,95,99,104,114,122\} \\
N_{100} & = & \{24,41,57,58,59,70,100,101,103,111,119\} \\
N_{101} & = & \{47,76,101,111,118\} \\
N_{102} & = & \{1,3,8,9,14,61,69,85,94,102,112,114\} \\
N_{103} & = & \{48,60,76,78,89,100,101,103,106,109\} \\
N_{104} & = & \{19,29,30,34,73,86,104\} \\
N_{105} & = & \{16,23,25,34,73,96,97,98,105,111,112\} \\
N_{106} & = & \{14,25,30,38,44,45,66,79,83,89,90,101,104,106,114,120\} \\
N_{107} & = & \{5,16,18,22,58,65,101,104,107\} \\
N_{108} & = & \{2,5,15,27,33,44,58,66,108,110,114,121\} \\
N_{109} & = & \{21,23,31,44,46,48,57,91,105,109,125\} \\
N_{110} & = & \{9,53,68,83,95,110,112,115,125\} \\
N_{111} & = & \{10,32,58,71,97,111,112,113,115,122\} \\
N_{112} & = & \{30,32,38,58,73,112,123,124\} \\
N_{113} & = & \{11,57,58,59,76,79,93,102,110,113,116\} \\
N_{114} & = & \{3,19,32,46,64,67,94,114\} \\
N_{115} & = & \{1,29,32,41,66,87,115\} \\
N_{116} & = & \{15,20,23,30,64,69,72,81,84,89,95,101,109,116,119,125\} \\
N_{117} & = & \{16,51,57,65,66,75,94,115,117\} \\
N_{118} & = & \{25,27,30,41,50,77,90,92,112,118\} \\
N_{119} & = & \{9,10,26,63,64,89,93,119\} \\
N_{120} & = & \{34,36,47,60,61,62,64,72,74,86,87,97,98,100,102,103,114,120,122\} \\
N_{121} & = & \{10,42,49,67,69,77,79,94,100,114,121\} \\
N_{122} & = & \{7,12,20,33,67,81,85,88,90,91,99,107,116,121,122,123,124\} \\
N_{123} & = & \{14,39,44,51,59,61,62,75,89,92,106,116,119,123\} \\
N_{124} & = & \{2,8,13,15,32,36,39,41,51,61,80,83,100,124\} \\
N_{125} & = & \{10,15,16,24,32,33,74,121,125\}
\end{array} \right.
$$

$$
\mathscr{A}_2 = \left\{ \begin{array}{lcl}
N_1 & = & \{1,21,25,34,36,40,60,80,105,124\} \\
N_2 & = & \{2,6,47,54,83,107,112\} \\
N_3 & = & \{2,3,39,62,84,102,106,120\} \\
N_4 & = & \{4,25,30,34,43,55,57,61,78,105\} \\
N_5 & = & \{2,5,6,15,21,22,34,41,49,60,62,63,69,83,102,106,115\} \\
N_6 & = & \{6,18,52,54,103,106\} \\
N_7 & = & \{7,8,21,24,27,40,41,54,62,90,104,111\} \\
N_8 & = & \{8,14,36,52,65,74,98,103\} \\
N_9 & = & \{7,9,26,35,47,52,82,86,88,91,95,105,120,122\} \\
N_{10} & = & \{10,13,53,60,75,95,118\} \\
N_{11} & = & \{8,11,22,37,48,52,76,99,116,118,120\} \\
N_{12} & = & \{3,8,12,17,55,59,69,84,108,114\} \\
N_{13} & = & \{13,14,40,41,50,55,62,68,70,74,86,94,98,100,105,107,109,120\} \\
N_{14} & = & \{14,30,33,48,54,58,59,79,86,90,103,120\} \\
N_{15} & = & \{15,26,29,38,47,51,76,87,107,115,118\} \\
N_{16} & = & \{6,16,23,32,37,55,112\} \\
N_{17} & = & \{17,31,37,53,62,63,65,110,120,121,123,125\} \\
N_{18} & = & \{3,15,18,25,27,38,40,45,55,82,94,103,110,117,120\} \\
N_{19} & = & \{19,61,79,85,114\} \\
N_{20} & = & \{7,9,14,16,20,27,50,65,73,74,81,83,87,95,104,108,119\} \\
N_{21} & = & \{9,19,21,40,67,76,84,109,116\} \\
N_{22} & = & \{22,38,47,51,62,80,102,113\} \\
N_{23} & = & \{15,18,23,29,35,46,59,78\} \\
N_{24} & = & \{24,38,45,73,74,78,79,108\} \\
N_{25} & = & \{25,27,29,44,67,84,90,93,101,109,113,114,116\} \\
N_{26} & = & \{7,26,30,71,87,98,120\} \\
N_{27} & = & \{2,25,27,31,46,47,82,95,97\} \\
N_{28} & = & \{6,28,35,37,49,69,72,85,112,116\} \\
N_{29} & = & \{7,17,21,29,41,46,48,54,56,66,67,73,79,118,121\} \\
N_{30} & = & \{11,15,30,31,34,55,69,124\} \\
N_{31} & = & \{3,16,18,30,31,35,36,39,43,48,68,69,72,84,96,112\} \\
N_{32} & = & \{10,13,14,31,32,55,68,80,94,97,105,125\} \\
N_{33} & = & \{3,16,21,28,33,43,48,64,86,108\} \\
N_{34} & = & \{23,27,30,34,37,46,48,63,67,86,94,116\} \\
N_{35} & = & \{9,19,35,78,90,108,123\} \\
N_{36} & = & \{14,24,36,37,74,102,105,114,121\} \\
N_{37} & = & \{37,68,76,85,88,92,93,107,118\} \\
N_{38} & = & \{37,38,43,45,50,60,73,92,113\} \\
N_{39} & = & \{2,24,30,39,40,78,82,93,102,118\} \\
N_{40} & = & \{10,11,40,58,78,94,95,118\} \\
N_{41} & = & \{6,14,22,23,26,28,41,48,56,70,86,94,96,99,107\} \\
N_{42} & = & \{34,41,42,72,76,80,89,103,104,109,118\} \\
N_{43} & = & \{8,43,53,57,76,81,88,89,94,124\} \\
N_{44} & = & \{3,44,63,76,81,102,103\} \\
N_{45} & = & \{11,14,17,21,44,45,55,75,122\} \\
N_{46} & = & \{5,31,35,38,46,49,56,60,65,74,82,87\} \\
N_{47} & = & \{23,26,28,33,38,47,67,68,86,92,96,121,124\} \\
N_{48} & = & \{48,63,109\} \\
N_{49} & = & \{4,6,23,30,36,37,49,61,67,68,71\} \\
N_{50} & = & \{4,7,14,17,28,30,41,45,50,51,53,54,56,72,86,113,114,115,122\} \\
N_{51} & = & \{15,16,21,25,31,47,50,51,62,66,78,109,114,121\} \\
N_{52} & = & \{13,23,52,63,65,82,90,99,102,105\} \\
N_{53} & = & \{25,45,48,53,76,88,96\} \\
N_{54} & = & \{9,29,54,57,67,73,85,92\} \\
N_{55} & = & \{11,14,16,52,53,55,70,78,98,107,109\} \\
N_{56} & = & \{15,36,45,55,56,69,72,79,88,98,108,119\} \\
N_{57} & = & \{2,19,36,57,76,90,91,120,121\} \\
N_{58} & = & \{12,32,39,45,58,94,110\} \\
N_{59} & = & \{7,24,34,59,62,72,92,95,96,119,124\} \\
N_{60} & = & \{6,19,53,60,74,86\} \\
N_{61} & = & \{4,7,9,10,17,26,59,61,66\} \\
N_{62} & = & \{15,20,26,27,43,50,55,58,62,65,69,71,77,114,115,119\} \\
N_{63} & = & \{6,8,11,34,41,63,80,84\} \\
N_{64} & = & \{38,61,64,68,70,84,107\}
\end{array} \right.
$$

$$\mathscr{A}_2 = \left\{ \begin{array}{lcl}
N_{65} & = & \{1,24,25,32,33,42,44,48,50,65,70,85,91,92,105\} \\
N_{66} & = & \{2,11,14,25,41,49,55,66,76,79\} \\
N_{67} & = & \{33,41,67,77,91,99,108,115\} \\
N_{68} & = & \{8,26,44,60,68,75,81,87,89,92,109,115,120\} \\
N_{69} & = & \{8,30,52,54,69,75,110\} \\
N_{70} & = & \{3,4,9,33,49,54,57,70,79,91,97,106,116,118,125\} \\
N_{71} & = & \{4,8,68,70,71,76,86,102,118,122\} \\
N_{72} & = & \{38,52,68,72,91,98,104\} \\
N_{73} & = & \{8,23,65,73,84,105,113,125\} \\
N_{74} & = & \{18,30,63,69,74,78,80,95\} \\
N_{75} & = & \{3,29,61,75,77,78,98,100,106,112,114\} \\
N_{76} & = & \{45,56,64,67,74,76,83,94,115,124\} \\
N_{77} & = & \{30,34,40,44,60,70,77,90,94,96,105,122\} \\
N_{78} & = & \{3,9,58,66,76,77,78,84\} \\
N_{79} & = & \{21,33,60,74,79,84,93,107,113\} \\
N_{80} & = & \{10,16,37,58,65,80,99,119,123,125\} \\
N_{81} & = & \{3,41,42,56,58,81,88,96\} \\
N_{82} & = & \{25,40,43,52,82,83,107,122,125\} \\
N_{83} & = & \{45,60,65,83,90,101,105,114,118\} \\
N_{84} & = & \{8,12,23,27,38,48,67,83,84,90,109,119,120\} \\
N_{85} & = & \{2,3,11,13,43,71,74,76,85,102\} \\
N_{86} & = & \{29,86,105,121\} \\
N_{87} & = & \{5,11,25,35,36,50,63,87,90,100,101,119,120\} \\
N_{88} & = & \{2,17,33,39,58,68,79,85,88,93,101,117\} \\
N_{89} & = & \{44,49,50,60,89,91,117\} \\
N_{90} & = & \{6,22,32,34,35,37,52,64,87,90,92,94,111,116,118\} \\
N_{91} & = & \{9,15,16,20,31,52,63,66,71,81,91,96,114\} \\
N_{92} & = & \{4,7,9,37,41,46,52,92,93,122\} \\
N_{93} & = & \{13,66,84,93,101,125\} \\
N_{94} & = & \{11,14,44,51,65,80,94,97,113,118\} \\
N_{95} & = & \{2,7,28,46,55,75,94,95,106\}
\end{array} \right.
\begin{array}{lcl}
N_{96} & = & \{20,21,33,63,64,68,96,98,119\} \\
N_{97} & = & \{7,23,25,26,67,97,103,117,120\} \\
N_{98} & = & \{5,15,16,17,24,31,32,56,67,72,86,98,110,114,117,122\} \\
N_{99} & = & \{6,23,29,30,99,106,114,116,121,125\} \\
N_{100} & = & \{20,24,32,52,53,77,100\} \\
N_{101} & = & \{38,40,62,69,75,77,82,83,97,101\} \\
N_{102} & = & \{6,19,21,32,35,46,47,66,78,97,98,101,102,113\} \\
N_{103} & = & \{48,65,67,103\} \\
N_{104} & = & \{11,27,38,47,72,80,104,105,122\} \\
N_{105} & = & \{31,38,39,45,53,85,86,96,98,101,105\} \\
N_{106} & = & \{5,15,19,36,42,73,81,91,100,104,106\} \\
N_{107} & = & \{3,24,25,52,54,62,63,66,79,84,85,96,107\} \\
N_{108} & = & \{6,9,11,35,71,89,99,103,108\} \\
N_{109} & = & \{12,18,27,29,56,61,62,109,111,112,116,117\} \\
N_{110} & = & \{12,21,28,44,66,70,73,87,96,100,105,110,116\} \\
N_{111} & = & \{1,30,38,51,76,83,92,101,110,111\} \\
N_{112} & = & \{18,20,47,86,89,91,104,112\} \\
N_{113} & = & \{14,19,42,58,62,80,92,97,113\} \\
N_{114} & = & \{4,6,56,114\} \\
N_{115} & = & \{13,21,26,30,32,55,58,62,66,72,78,82,93,97,98,104,115,117\} \\
N_{116} & = & \{10,34,39,46,51,80,81,91,103,104,109,116\} \\
N_{117} & = & \{12,57,79,97,100,113,117,123\} \\
N_{118} & = & \{4,7,26,36,46,69,118,121\} \\
N_{119} & = & \{5,13,29,31,33,50,53,65,73,76,82,106,114,119\} \\
N_{120} & = & \{25,28,30,46,53,70,71,107,120\} \\
N_{121} & = & \{4,12,29,34,61,62,63,88,97,112,121\} \\
N_{122} & = & \{15,21,37,43,49,51,56,87,88,99,104,111,117,120,122\} \\
N_{123} & = & \{2,49,59,65,80,82,89,120,123\} \\
N_{124} & = & \{13,22,45,62,64,68,71,73,85,87,110,124\} \\
N_{125} & = & \{2,35,40,49,94,95,125\}
\end{array}$$

$$\mathscr{A}_3 = \left\{ \begin{array}{lcl}
N_{1} & = & \{1,4,63,74,77,80\} \\
N_{2} & = & \{2,9,12,18,21,100,105,125\} \\
N_{3} & = & \{3,12,16,22,51,54,62,64,96,97,100,104,105,111,116\} \\
N_{4} & = & \{4,18,21,34,41,90\} \\
N_{5} & = & \{5,72,93,98\} \\
N_{6} & = & \{6,55,78,98,112,123\} \\
N_{7} & = & \{7,14,31,35,58,78,116,120\} \\
N_{8} & = & \{2,8,11,24,26,28,32,34,54,60,84,102\} \\
N_{9} & = & \{8,9,17,41,58,77,78,102,105\} \\
N_{10} & = & \{5,7,10,15,20,45,53,56,70,73,78,90,92,101,104,108,111,114,116\} \\
N_{11} & = & \{4,11,17,22,66,68,85,102,105,116,125\} \\
N_{12} & = & \{12,19,26,29,36,54,65,90,96,104\} \\
N_{13} & = & \{13,35,39,56,70,73,83,109,118,124\} \\
N_{14} & = & \{14,15,22,40,62,64,69,93,101,121\} \\
N_{15} & = & \{8,13,15,22,29,40,52,56,68,73,83,91,116\} \\
N_{16} & = & \{12,16,41,44,55,58,90,119\} \\
N_{17} & = & \{3,4,17,46,82,86,96\} \\
N_{18} & = & \{4,15,18,29,35,38,45,71,117\} \\
N_{19} & = & \{2,6,19,31,59,74,85,104,110,114\} \\
N_{20} & = & \{3,20,36,44,49,62,78,107,118\} \\
N_{21} & = & \{12,21,33,50,56,72,90,97,100,112,115\} \\
N_{22} & = & \{8,22,36,38,45,53,56,63,67,89,90,106,119\} \\
N_{23} & = & \{13,23,33,42,51,59,75,84,112,113\} \\
N_{24} & = & \{21,24,86,88,99,118\} \\
N_{25} & = & \{8,25,40,42,74,98,107\} \\
N_{26} & = & \{3,26,28,39,54,63,87,92,120,125\} \\
N_{27} & = & \{27,34,44,52,53,66,71,81,89,92,107,112\} \\
N_{28} & = & \{13,28\} \\
N_{29} & = & \{3,20,26,29,40,61,68,85,95,109,111\} \\
N_{30} & = & \{9,16,19,30,38,48,66,68,72,76,78,89,92,95,103,115\} \\
N_{31} & = & \{5,25,30,31,47,50,56,90,93,97,103\} \\
N_{32} & = & \{5,28,32,44,72,73,80,92,93,96,101,120\} \\
N_{33} & = & \{14,20,29,33,35,36,68,74,105,109,112\} \\
N_{34} & = & \{14,34,45,47,83,93,103\} \\
N_{35} & = & \{8,31,35,55,60,61,76\} \\
N_{36} & = & \{36,66,70,86,106,112\} \\
N_{37} & = & \{37,44,53,65,71,81,101,120\} \\
N_{38} & = & \{1,3,19,38,44,64,65,83,89,99,115\} \\
N_{39} & = & \{39,44,77,91,94,102,104\} \\
N_{40} & = & \{14,20,39,40,45,68,71,72\} \\
N_{41} & = & \{20,41,45,63,74,90,102,123\} \\
N_{42} & = & \{3,36,42,43,52,77,78,81,106,107\} \\
N_{43} & = & \{2,11,43,74,78,82,113,114\} \\
N_{44} & = & \{3,5,10,21,24,44,52,53,58,93,98,103,111\} \\
N_{45} & = & \{4,12,45,73,82,120,122\} \\
N_{46} & = & \{4,5,15,30,31,43,46,60,73,76,89,106,116,117,119,121,124\} \\
N_{47} & = & \{4,27,32,41,47,53,70,80\} \\
N_{48} & = & \{1,3,48,85,94,106,107\} \\
N_{49} & = & \{14,25,30,36,49,53,90,115,119\} \\
N_{50} & = & \{3,14,30,49,50,60,67,99,106,107,112,117\} \\
N_{51} & = & \{7,11,27,47,51,61,62,83,90,92,109\} \\
N_{52} & = & \{3,13,33,45,52,83,88,90,97,103\} \\
N_{53} & = & \{42,46,53,77,84,98,99,101,105\} \\
N_{54} & = & \{17,18,32,43,46,54,64,65,107\} \\
N_{55} & = & \{6,17,27,29,35,51,55,79,105,115\} \\
N_{56} & = & \{2,38,41,46,56,61,92,119\} \\
N_{57} & = & \{3,14,24,57,79,84,96,102,112\} \\
N_{58} & = & \{2,18,24,42,52,58,59,69,90,102,107\} \\
N_{59} & = & \{2,25,31,47,52,59,89,105,113\} \\
N_{60} & = & \{6,15,29,32,60\} \\
N_{61} & = & \{12,25,33,61,88,92,93,123\} \\
N_{62} & = & \{9,17,34,38,51,62,98,99\} \\
N_{63} & = & \{1,19,34,46,47,63,66,67,101,104,124\}
\end{array} \right.
\begin{array}{lcl}
N_{64} & = & \{64,112,116,125\} \\
N_{65} & = & \{6,7,41,60,65,106,111,112,114\} \\
N_{66} & = & \{50,66,83,97,100\} \\
N_{67} & = & \{18,22,24,31,32,35,48,58,67,77,83,95,102,104,125\} \\
N_{68} & = & \{6,16,41,43,45,59,61,68,101,104,124\} \\
N_{69} & = & \{4,20,32,36,37,38,50,69,76,86,121,125\} \\
N_{70} & = & \{9,11,12,13,53,70,74,80,87,96,100,113,116,118,124\} \\
N_{71} & = & \{1,9,27,29,59,63,65,71,77,84,91,97,103\} \\
N_{72} & = & \{31,46,68,72,125\} \\
N_{73} & = & \{24,38,55,66,70,73,78,88,98,111\} \\
N_{74} & = & \{24,35,45,48,65,68,69,74,86,105,110,117\} \\
N_{75} & = & \{27,31,38,49,53,60,74,75,77,92,106,123\} \\
N_{76} & = & \{2,11,31,33,58,61,63,71,76,90,92,95,123,125\} \\
N_{77} & = & \{36,62,77,80,96,124,125\} \\
N_{78} & = & \{5,31,40,61,73,78,84,94\} \\
N_{79} & = & \{13,24,26,45,51,57,66,76,79,84,99,109\} \\
N_{80} & = & \{2,5,11,17,29,40,51,57,72,80,92,99,122,123\} \\
N_{81} & = & \{29,41,45,81,89,91,113\} \\
N_{82} & = & \{12,17,38,46,50,64,81,82,97,105,110\} \\
N_{83} & = & \{29,42,54,83,106\} \\
N_{84} & = & \{25,26,27,29,30,31,53,61,84,113,115,118,121\} \\
N_{85} & = & \{40,54,69,85\} \\
N_{86} & = & \{40,46,53,56,63,67,73,77,86,100,112,123\} \\
N_{87} & = & \{10,19,24,27,33,37,38,41,52,62,82,87,102,110,116,117\} \\
N_{88} & = & \{4,36,73,88,123\} \\
N_{89} & = & \{3,4,31,89,101,112\} \\
N_{90} & = & \{4,5,6,9,23,33,69,72,90,98,110\} \\
N_{91} & = & \{10,31,83,88,91,100,107,113\} \\
N_{92} & = & \{19,22,35,54,55,67,92,104,107,108,120,122\} \\
N_{93} & = & \{5,10,22,28,32,71,84,93,96,115,121\} \\
N_{94} & = & \{54,74,76,91,94\} \\
N_{95} & = & \{45,46,47,68,79,82,83,84,92,95,99\} \\
N_{96} & = & \{2,23,32,57,69,70,72,76,96,98,115,121\} \\
N_{97} & = & \{15,16,25,74,78,97,99,116,123\} \\
N_{98} & = & \{7,31,33,50,62,82,96,98,102,111,116\} \\
N_{99} & = & \{6,8,45,53,62,65,83,99,105,108,109\} \\
N_{100} & = & \{7,28,35,54,65,70,81,83,100,105,106\} \\
N_{101} & = & \{22,43,48,51,75,76,87,94,101,105,110,120\} \\
N_{102} & = & \{6,8,49,52,78,85,101,102,104,107,117\} \\
N_{103} & = & \{5,7,24,31,34,39,59,61,83,85,96,103,109,111,124\} \\
N_{104} & = & \{8,36,53,65,86,95,103,104,110,115\} \\
N_{105} & = & \{6,41,44,56,78,79,86,90,94,105,114,124\} \\
N_{106} & = & \{17,39,56,74,91,95,106,114\} \\
N_{107} & = & \{12,16,32,50,54,56,107,123\} \\
N_{108} & = & \{3,13,50,53,60,75,83,91,97,108,117,121\} \\
N_{109} & = & \{4,16,26,29,32,34,43,65,69,74,93,94,97,107,109,114\} \\
N_{110} & = & \{20,21,22,38,53,81,96,110,111,113,114\} \\
N_{111} & = & \{6,37,44,54,101,111\} \\
N_{112} & = & \{12,17,22,24,90,101,112,119\} \\
N_{113} & = & \{1,13,15,21,24,63,65,73,81,89,91,96,113,118\} \\
N_{114} & = & \{35,43,47,69,82,113,114\} \\
N_{115} & = & \{4,9,48,51,65,66,69,70,73,90,98,115,119\} \\
N_{116} & = & \{16,36,37,40,47,53,56,79,80,82,98,103,104,106,111,116,122,123\} \\
N_{117} & = & \{46,78,96,109,117,118\} \\
N_{118} & = & \{34,47,71,96,111,115,118,120\} \\
N_{119} & = & \{17,49,57,72,75,101,116,119\} \\
N_{120} & = & \{6,25,35,45,60,114,120,121\} \\
N_{121} & = & \{15,20,30,46,54,65,77,94,121,123\} \\
N_{122} & = & \{11,13,14,16,33,46,67,96,104,122\} \\
N_{123} & = & \{30,34,39,56,62,63,69,111,116,123\} \\
N_{124} & = & \{34,47,50,86,90,114,124\} \\
N_{125} & = & \{33,36,70,73,125\}
\end{array}$$

$$\mathscr{A}_4 = \left\{ \begin{array}{lllllll}
N_{1} & = & \{1,4,22,28,30,32,41,45,61,62,68,79,92,105,114,116\} & & N_{64} & = & \{7,9,17,26,38,43,47,64,66,68,91,98,108,114\} \\
N_{2} & = & \{2,28,31,42,48,64,72,77,79,81,88,92,122\} & & N_{65} & = & \{3,19,29,39,45,55,64,65,102,109\} \\
N_{3} & = & \{3,12,42,49,50\} & & N_{66} & = & \{2,5,10,20,33,40,42,64,66,99,106,107,108,112,121\} \\
N_{4} & = & \{4,7,45,51,54,55,63,79,81,111\} & & N_{67} & = & \{1,50,58,67,82,91,107,109\} \\
N_{5} & = & \{5,8,24,25,30,38,48,110\} & & N_{68} & = & \{14,32,46,68,95,96\} \\
N_{6} & = & \{3,4,6,34,41,51,54,58,61,63,76,106,122\} & & N_{69} & = & \{26,43,69,78,83,90,101\} \\
N_{7} & = & \{4,7,13,34,61,76,84,94,111\} & & N_{70} & = & \{2,12,14,28,30,42,50,70,76,85,106,114,118,119\} \\
N_{8} & = & \{2,8,27,60,79,85,95,101\} & & N_{71} & = & \{2,5,9,35,43,52,61,71,101,120\} \\
N_{9} & = & \{9,10,13,15,21,28,31,55,56,65,85,111,124\} & & N_{72} & = & \{5,12,13,38,57,61,65,72,76,105,107,119\} \\
N_{10} & = & \{1,10,23,31,54,107,113,123\} & & N_{73} & = & \{9,16,36,72,73,81,86,109,116\} \\
N_{11} & = & \{11,17,29,40,43,46,53,65,73,96,104,111,117,121,125\} & & N_{74} & = & \{9,32,37,53,56,58,60,68,74,78,79,107,114,121,123\} \\
N_{12} & = & \{8,12,74,90,106,107,116\} & & N_{75} & = & \{10,14,38,42,51,75,98,108,116,117\} \\
N_{13} & = & \{13,24,31,44,53,93\} & & N_{76} & = & \{8,15,27,76,95,102,120,125\} \\
N_{14} & = & \{13,14,21,115,122\} & & N_{77} & = & \{1,21,29,43,62,72,73,77,91,92,103,108,116,121\} \\
N_{15} & = & \{6,14,15,28,33,45,51,84,90,105\} & & N_{78} & = & \{4,8,40,49,50,70,78,84,105,119\} \\
N_{16} & = & \{6,9,16,75,109,123\} & & N_{79} & = & \{7,9,39,51,54,79,88,90,100,112,118\} \\
N_{17} & = & \{2,17,22,31,38,59,115\} & & N_{80} & = & \{37,79,80,90,102,113,115,116,124\} \\
N_{18} & = & \{18,70,93,95,117\} & & N_{81} & = & \{15,26,40,61,62,67,68,69,81,93,99,109\} \\
N_{19} & = & \{11,19,23,34,45,56,65,93,112\} & & N_{82} & = & \{5,25,35,48,50,59,75,81,82,95,113,124\} \\
N_{20} & = & \{20,23,37,66,95,104,107,116,120\} & & N_{83} & = & \{2,5,21,36,57,58,83,92,99,114,125\} \\
N_{21} & = & \{7,14,17,21,31,44,54,62,69,111,113,121\} & & N_{84} & = & \{24,40,84,85,108,117,124\} \\
N_{22} & = & \{5,6,15,22,33,36,45,69,79\} & & N_{85} & = & \{5,25,32,38,49,54,85,90,122\} \\
N_{23} & = & \{20,23,58,67,76,81,84,113,119\} & & N_{86} & = & \{13,21,49,70,72,86,95,96,100,105,106,107,108,109,118\} \\
N_{24} & = & \{24,37,40,43,48,75,86,99,107,118\} & & N_{87} & = & \{10,17,26,28,40,44,65,71,77,87,89,102,115,116,117,121\} \\
N_{25} & = & \{11,25,29,50,60,88,91,111,125\} & & N_{88} & = & \{12,13,23,54,59,60,67,88,100,104\} \\
N_{26} & = & \{15,18,21,26,28,64,69,79,83,104,119\} & & N_{89} & = & \{89\} \\
N_{27} & = & \{4,12,27,32,33,36,76,78,90,93,100\} & & N_{90} & = & \{44,74,90,97,120,123\} \\
N_{28} & = & \{24,28,45,53,122\} & & N_{91} & = & \{6,25,41,48,49,63,72,86,91,108,112,114,121\} \\
N_{29} & = & \{25,29,39,54,69,78,99\} & & N_{92} & = & \{17,20,38,62,65,91,92,116\} \\
N_{30} & = & \{3,17,20,30,36,44,50,71,87,93,94,96\} & & N_{93} & = & \{2,8,12,38,64,65,90,93,100,102,104,111\} \\
N_{31} & = & \{8,15,20,22,31,41,56,59,61,62,67,78,79,82,102,107,122,124\} & & N_{94} & = & \{19,44,59,68,74,88,94,107,109\} \\
N_{32} & = & \{3,32,62,68,73,75,84,103,113\} & & N_{95} & = & \{18,28,42,50,57,62,70,86,95\} \\
N_{33} & = & \{13,16,19,20,26,29,30,31,32,33,43,61,82,85,89,99,123\} & & N_{96} & = & \{5,35,41,43,55,71,77,93,96,100,124\} \\
N_{34} & = & \{4,12,25,34,56,58,71,79,98\} & & N_{97} & = & \{31,34,42,44,47,54,67,71,74,76,93,97,108\} \\
N_{35} & = & \{1,9,21,24,35,41,52,80,88,90,104\} & & N_{98} & = & \{8,22,23,30,35,36,63,84,98,106,111,121\} \\
N_{36} & = & \{3,7,12,20,36,38,84,116\} & & N_{99} & = & \{26,28,39,66,78,91,99,105,124\} \\
N_{37} & = & \{8,22,31,37,50,54,68,72,86,88,113,117,121\} & & N_{100} & = & \{20,48,54,84,100,115\} \\
N_{38} & = & \{8,13,14,31,38,42,46,92,103,106,107,116\} & & N_{101} & = & \{13,14,37,89,100,101,120\} \\
N_{39} & = & \{5,16,39,72,80,82,108,109,111\} & & N_{102} & = & \{7,12,30,42,45,57,64,70,79,85,87,94,102,104,117\} \\
N_{40} & = & \{16,39,40,41,55,61,67,70,73,81,85,90,99,111\} & & N_{103} & = & \{2,39,55,70,74,85,103\} \\
N_{41} & = & \{29,41,49,69,88,97,110,115,121\} & & N_{104} & = & \{34,36,51,56,60,65,77,91,101,104,109,120\} \\
N_{42} & = & \{1,9,11,14,38,42,50,54,55,58,67,75,86,92,117\} & & N_{105} & = & \{5,60,65,78,88,90,105,106,110,115\} \\
N_{43} & = & \{3,9,40,43,92\} & & N_{106} & = & \{21,26,29,38,53,54,56,59,64,65,71,96,98,102,106\} \\
N_{44} & = & \{1,44,50,58,62,82,106,124\} & & N_{107} & = & \{12,17,31,55,61,70,73,96,101,107,116\} \\
N_{45} & = & \{1,12,29,30,45,52,75,101\} & & N_{108} & = & \{3,38,42,49,65,75,103,105,107,108,120\} \\
N_{46} & = & \{7,46,55,70,77,79,92,97,114,117,118\} & & N_{109} & = & \{5,8,47,63,64,104,109,113\} \\
N_{47} & = & \{2,4,7,22,23,24,47,97\} & & N_{110} & = & \{21,22,30,33,48,57,61,107,110\} \\
N_{48} & = & \{10,35,48,63,72,85,98,105,107,113,120,122\} & & N_{111} & = & \{4,7,8,15,18,19,51,81,95,111,112,117\} \\
N_{49} & = & \{3,16,20,39,49,65,73,77,106\} & & N_{112} & = & \{39,105,107,112\} \\
N_{50} & = & \{21,37,40,47,49,50,70,89,91,93,94,99,104,111\} & & N_{113} & = & \{2,5,14,16,41,48,56,59,103,113,119\} \\
N_{51} & = & \{15,22,29,42,51,67,71,91,93,97,103,112,118,123\} & & N_{114} & = & \{9,52,64,69,74,91,104,105,113,114,124,125\} \\
N_{52} & = & \{11,28,33,52,101,105,106,111\} & & N_{115} & = & \{14,34,53,61,78,82,105,106,115\} \\
N_{53} & = & \{26,41,50,53,74,76,81,100,101,109\} & & N_{116} & = & \{5,6,7,24,71,73,86,88,91,92,96,116\} \\
N_{54} & = & \{2,12,33,36,54,82,109,118\} & & N_{117} & = & \{3,15,35,47,48,71,88,117,122\} \\
N_{55} & = & \{3,8,9,36,49,51,55,58,59,89,92,102,104,113\} & & N_{118} & = & \{1,6,26,53,82,86,87,94,109,118\} \\
N_{56} & = & \{27,29,38,40,50,56,69,72,73,87,101,103,104,107\} & & N_{119} & = & \{2,18,30,33,61,86,119\} \\
N_{57} & = & \{1,33,38,44,52,57,62,63,65,67,76,109,114\} & & N_{120} & = & \{4,54,65,78,86,87,94,100,102,113,114,117,120\} \\
N_{58} & = & \{3,21,30,37,44,58,62,70,83,92,105,106\} & & N_{121} & = & \{8,34,44,52,76,87,114,117,121,124\} \\
N_{59} & = & \{5,15,28,43,47,59,60,85,97,104,108,110,114,125\} & & N_{122} & = & \{13,29,41,63,72,78,79,83,85,90,93,106,122\} \\
N_{60} & = & \{7,12,16,46,53,55,60,95,108,114\} & & N_{123} & = & \{12,19,30,73,84,90,112,123\} \\
N_{61} & = & \{10,12,39,43,50,61,67,72,95,119\} & & N_{124} & = & \{6,12,18,25,33,54,64,69,94,101,105,124\} \\
N_{62} & = & \{6,41,62,92,96,121,122\} & & N_{125} & = & \{12,17,56,66,78,91,99,101,103,122,125\} \\
N_{63} & = & \{48,57,62,63\} & & & &
\end{array} \right.$$

$$\mathscr{A}_5 = \left\{ \begin{array}{lllllll}
N_{1} & = & \{1,14,24,25,29,37,69,75,79,94,97,118\} & & N_{33} & = & \{25,33,93,103,108,123\} \\
N_{2} & = & \{2,27,53,92,97,104,108,114,124\} & & N_{34} & = & \{4,12,20,28,34,36,55,62,73,122\} \\
N_{3} & = & \{3,11,13,20,90,93,95,112\} & & N_{35} & = & \{31,35,37,44,46,51,57,68,76,108,123\} \\
N_{4} & = & \{4,22,35,55,65,68,93,101,107,109,117,118\} & & N_{36} & = & \{7,12,35,36,42,44,66,67,76,85,91\} \\
N_{5} & = & \{5,16,17,42,55,91,107,115\} & & N_{37} & = & \{6,37,41,95,120,123\} \\
N_{6} & = & \{6,17,24,42,85,89,120,121\} & & N_{38} & = & \{13,17,28,30,38,45,59,66,94,103,118\} \\
N_{7} & = & \{7,28,41,66,82,88,94,96\} & & N_{39} & = & \{15,16,39,46,47,51,63,70\} \\
N_{8} & = & \{6,8,17,41,63,73,78,92,104,105\} & & N_{40} & = & \{40,41,51,58,66,69,94,100,117\} \\
N_{9} & = & \{9,10,18,53,60,68,84,89,113,115\} & & N_{41} & = & \{1,22,25,41,60,63,70,86,98,107\} \\
N_{10} & = & \{1,10,15,38,58,65,78,80,85,95,114,115,125\} & & N_{42} & = & \{5,10,20,42,48,51,67,110,117\} \\
N_{11} & = & \{3,4,11,22,27,39,44,46,60,62,78,85,95,102,107,110,111\} & & N_{43} & = & \{9,11,29,35,40,43,91,96,97,105,106,122\} \\
N_{12} & = & \{9,11,12,17,39,45,47,58,62,70,82,91\} & & N_{44} & = & \{1,5,36,42,44,70,76,77,106,112\} \\
N_{13} & = & \{13,14,53,56,65,67,115\} & & N_{45} & = & \{16,35,41,45,59,75,102,119,122,125\} \\
N_{14} & = & \{14,27,32,39,40,42,58,85,95,107,109,110\} & & N_{46} & = & \{4,19,25,45,46,59,63,77,83,86,88,95,97,105,109\} \\
N_{15} & = & \{15,31,44,65,68\} & & N_{47} & = & \{14,17,26,28,47,54,58,80,97,123\} \\
N_{16} & = & \{9,16,18,19,27,38,50,59,74,76,83,98,115,119,121\} & & N_{48} & = & \{4,5,48,58,60,82,90,122\} \\
N_{17} & = & \{17,57,69,77,102,107,120\} & & N_{49} & = & \{9,23,24,28,30,49,57,59,78,100,105,108\} \\
N_{18} & = & \{11,18,63,82,99,105,114,119,123\} & & N_{50} & = & \{4,17,27,30,50,51,56,78,96,113,119\} \\
N_{19} & = & \{1,12,19,44,49,89,93,101,113\} & & N_{51} & = & \{7,11,48,51,68,82,88,105,108,115,121\} \\
N_{20} & = & \{5,14,18,20,27,50,63,79,102\} & & N_{52} & = & \{38,52,76,84,87,88\} \\
N_{21} & = & \{1,21,23,36,61,67\} & & N_{53} & = & \{21,28,42,43,44,53,69,83,87,105\} \\
N_{22} & = & \{3,22,33,65,66,71,101,105\} & & N_{54} & = & \{1,25,28,40,52,54,81,97\} \\
N_{23} & = & \{8,14,21,23,32,38,42,46,51,62,64,70,79,97,104,106,113,114\} & & N_{55} & = & \{4,9,53,55,70,73,82,112\} \\
N_{24} & = & \{13,22,24,53,63,72,90,95,97,104,109,120\} & & N_{56} & = & \{7,9,23,46,52,56,75,80,95,101,106,110,111\} \\
N_{25} & = & \{7,11,25,26,31,42,51,55,65,68,93,101,114,122\} & & N_{57} & = & \{11,57,59,66,89,92,97,109,119,121\} \\
N_{26} & = & \{25,26,31,38,46,48,111\} & & N_{58} & = & \{32,54,58,71,119\} \\
N_{27} & = & \{27,31,40,46,50,63,64,84,99,104,113,114,116,119\} & & N_{59} & = & \{5,11,13,14,42,45,59,86,106,122,124\} \\
N_{28} & = & \{8,28,44,46,48,62,65,66,69,92,96\} & & N_{60} & = & \{7,12,27,33,39,60,79,80,95,106,115\} \\
N_{29} & = & \{6,29,41,57,59,60,61,67,70,72,93,99,113,125\} & & N_{61} & = & \{3,7,12,31,35,55,61,77,98,118,122\} \\
N_{30} & = & \{9,26,28,30,33,42,49,74,77,86,101,121\} & & N_{62} & = & \{3,6,16,21,26,58,62,73,75,76,99,124\} \\
N_{31} & = & \{28,31,35,46,54,97,101,115,121\} & & N_{63} & = & \{44,46,59,63,77,84\} \\
N_{32} & = & \{17,21,32,35,36,76,84,98,109\} & & N_{64} & = & \{29,40,64,94,108,112,125\}
\end{array} \right.$$

$$\mathscr{A}_5 = \left\{ \begin{array}{lll}
N_{65} & = & \{12,20,22,26,65,72,77,96,105,106,109,112\} \\
N_{66} & = & \{25,43,66,73,74,77,81,90,100,108\} \\
N_{67} & = & \{16,18,38,45,67,70,72,91,106,108\} \\
N_{68} & = & \{3,14,29,68,75,86,95,107,116\} \\
N_{69} & = & \{7,19,35,46,57,69,76,81,93,113,114\} \\
N_{70} & = & \{1,27,35,48,56,70,88,89,116,121,123\} \\
N_{71} & = & \{8,13,34,40,49,57,58,71,80,81,84,90,101,118\} \\
N_{72} & = & \{8,30,55,60,67,68,72,99,113\} \\
N_{73} & = & \{8,22,50,53,61,70,73,76,80,81,91,92,95,97,109\} \\
N_{74} & = & \{6,12,36,41,48,51,52,58,74,91,94,99,107,118\} \\
N_{75} & = & \{17,51,75,90,91,112,116\} \\
N_{76} & = & \{4,5,19,23,50,52,58,60,72,76,78,100,102\} \\
N_{77} & = & \{7,18,59,61,72,77,101,108,109,117,123,124\} \\
N_{78} & = & \{34,45,50,62,78,85,122\} \\
N_{79} & = & \{1,3,4,16,22,24,34,37,40,48,49,69,79,89,90,117,122\} \\
N_{80} & = & \{6,9,10,27,44,63,80,81,84,118\} \\
N_{81} & = & \{4,7,13,26,46,49,71,81,100,103,106,117,125\} \\
N_{82} & = & \{3,21,28,34,54,56,82,85,87,91,103,117\} \\
N_{83} & = & \{6,15,19,30,64,81,83,85,98,109\} \\
N_{84} & = & \{3,4,5,10,27,47,65,84,85,104,117\} \\
N_{85} & = & \{3,39,43,60,85,91,93,121,124\} \\
N_{86} & = & \{11,21,42,56,58,79,86,87,99\} \\
N_{87} & = & \{6,13,36,38,65,81,83,87,88,99,101,117,119\} \\
N_{88} & = & \{11,13,19,24,32,57,83,88,95,101,103,112\} \\
N_{89} & = & \{1,24,30,56,68,69,79,89\} \\
N_{90} & = & \{6,32,63,67,89,90,96,102,118,119\} \\
N_{91} & = & \{33,47,51,77,85,91,99,120,122,125\} \\
N_{92} & = & \{4,15,29,61,80,81,91,92,108,118,121\} \\
N_{93} & = & \{7,39,46,50,57,61,85,93,96,98\} \\
N_{94} & = & \{14,19,26,65,81,94,117\} \\
N_{95} & = & \{12,28,31,46,49,58,59,87,92,95,112,114\} \\
N_{96} & = & \{8,13,19,25,66,68,96,103,105,118\} \\
N_{97} & = & \{5,6,27,36,37,42,46,51,76,96,97,103,108,109\} \\
N_{98} & = & \{24,37,55,59,71,77,98,123\} \\
N_{99} & = & \{10,13,52,54,63,64,65,91,99\} \\
N_{100} & = & \{56,63,87,90,92,99,100\} \\
N_{101} & = & \{22,31,33,42,63,101,125\} \\
N_{102} & = & \{40,43,61,69,85,93,96,102,109,117\} \\
N_{103} & = & \{8,14,33,39,51,58,64,77,103,106,108,110,112\} \\
N_{104} & = & \{6,9,10,14,30,44,46,71,79,80,103,104,118\} \\
N_{105} & = & \{21,34,38,45,47,61,71,99,105,119\} \\
N_{106} & = & \{25,34,41,45,63,89,96,104,106\} \\
N_{107} & = & \{1,13,15,49,59,76,79,105,107\} \\
N_{108} & = & \{8,16,67,83,87,101,102,108\} \\
N_{109} & = & \{8,9,26,34,54,61,78,91,99,103,104,107,109,120\} \\
N_{110} & = & \{10,18,51,89,104,108,109,110,119\} \\
N_{111} & = & \{5,6,11,14,43,55,98,111\} \\
N_{112} & = & \{6,8,12,19,61,68,84,86,104,112,115,118,121\} \\
N_{113} & = & \{8,10,16,85,91,103,113,121\} \\
N_{114} & = & \{19,21,31,54,63,74,89,114,117,120,123,125\} \\
N_{115} & = & \{29,33,45,48,49,67,75,76,93,97,101,102,109,115\} \\
N_{116} & = & \{34,40,41,67,72,83,102,116\} \\
N_{117} & = & \{14,18,29,33,117\} \\
N_{118} & = & \{38,86,91,101,118,119\} \\
N_{119} & = & \{4,17,40,52,81,90,118,119\} \\
N_{120} & = & \{4,5,10,13,19,24,33,35,60,62,67,70,95,96,98,99,101,103,104,105,120\} \\
N_{121} & = & \{14,22,49,51,63,68,70,81,95,100,121\} \\
N_{122} & = & \{41,47,50,55,79,85,88,102,107,112,122\} \\
N_{123} & = & \{9,15,18,41,45,47,52,56,75,76,111,122,123\} \\
N_{124} & = & \{4,10,13,31,33,46,51,77,96,112,124\} \\
N_{125} & = & \{6,11,71,92,111,114,125\}
\end{array} \right.$$

$$\mathscr{A}_6 = \left\{ \begin{array}{lll}
N_{1} & = & \{1,13,29,76,86,117\} \\
N_{2} & = & \{2,17,30,51,52,58,62,63,68,72,85,113,114\} \\
N_{3} & = & \{3,35,45,49,56,83,112\} \\
N_{4} & = & \{4,19,21,38,43,60,67,71,94,102,121\} \\
N_{5} & = & \{1,5,15,24,35,36,49,52,64,65,99,112,119\} \\
N_{6} & = & \{2,3,5,6,14,18,23,47,53,94,97,98,106\} \\
N_{7} & = & \{5,6,7,54,57,80,83,85\} \\
N_{8} & = & \{8,32,49,65,73,98\} \\
N_{9} & = & \{9,14,18,26,30,84,85,98\} \\
N_{10} & = & \{10,29,68,71,92,94,98,103,108\} \\
N_{11} & = & \{3,11,22,23,32,42,52,75,123,124\} \\
N_{12} & = & \{12,15,38,44,48,63,80,81,98,125\} \\
N_{13} & = & \{2,13,18,21,76,82,85,88,105,123\} \\
N_{14} & = & \{3,14,112\} \\
N_{15} & = & \{10,12,15,46,77,82,108\} \\
N_{16} & = & \{7,16,21,42,49,53,56,68,71,85,105,115,116,124\} \\
N_{17} & = & \{17,18,27,36,55,67,71,99,112\} \\
N_{18} & = & \{7,18,22,24,30,98,100,112\} \\
N_{19} & = & \{17,19,27,31,32,42,83,84,86,101,108\} \\
N_{20} & = & \{7,15,19,20,25,30,58,70,74,87,94\} \\
N_{21} & = & \{21,34,39,41,62,65,81,83,88,95,106\} \\
N_{22} & = & \{22,29,30,31,33,82,105,122\} \\
N_{23} & = & \{7,23,24,36,40,70,75,78,94,102,109\} \\
N_{24} & = & \{1,6,8,15,18,22,24,52,59,60,62,112,121\} \\
N_{25} & = & \{14,23,25,42,65,71,72,82,98,105,107,108,114\} \\
N_{26} & = & \{26,33,44,80,81,82,91,96,115,124\} \\
N_{27} & = & \{27,39,67,72,73,80,92,109,111\} \\
N_{28} & = & \{28,52,77,94,96\} \\
N_{29} & = & \{19,29,38,42,66,84,96,99,111\} \\
N_{30} & = & \{5,8,23,26,28,30,36,49,57,76,84,107\} \\
N_{31} & = & \{1,25,31,53,56,95,98,104,105\} \\
N_{32} & = & \{13,32,39,47,51,70,83,92\} \\
N_{33} & = & \{19,33,35,36,41,53,63,103,124\} \\
N_{34} & = & \{7,18,34,72,77,88,101,104\} \\
N_{35} & = & \{17,18,21,35,41,57,62,74,79,90,101,111,114,125\} \\
N_{36} & = & \{3,20,22,28,36,39,68,69,88,111,116\} \\
N_{37} & = & \{1,5,32,37,40,50,54,56,75,93,116\} \\
N_{38} & = & \{13,20,32,36,38,79,88,90,97\} \\
N_{39} & = & \{6,39,44,57,73,103,121\} \\
N_{40} & = & \{40,45,49,64,75,78,80,87,101,103,107,114,119\} \\
N_{41} & = & \{3,4,6,24,37,41,70,84,86,93,112,114,121\} \\
N_{42} & = & \{14,26,36,42,73,86,103,110\} \\
N_{43} & = & \{8,22,31,39,43,59,64,82,88,92,94,113\} \\
N_{44} & = & \{1,44,49,52,65,67,82,84,88,97\} \\
N_{45} & = & \{23,45,95,98,105,123\} \\
N_{46} & = & \{46,66,71,75,81,104,118\} \\
N_{47} & = & \{7,16,19,36,42,47,53,96,102,114,115\} \\
N_{48} & = & \{17,28,48,55,108,124\} \\
N_{49} & = & \{41,42,49,52,57,61,71,74,76,81,92,97,107,109\} \\
N_{50} & = & \{1,50,51,59,70,86,91,112\} \\
N_{51} & = & \{1,10,20,34,39,51,56,77,107\} \\
N_{52} & = & \{28,29,52,59,68,80,94,108,109,118\} \\
N_{53} & = & \{14,15,44,53,57,64,78,96,123\} \\
N_{54} & = & \{37,54,61,67,84,110,122\} \\
N_{55} & = & \{16,35,55,75,76,77,80,84,97,116\} \\
N_{56} & = & \{7,9,20,26,36,42,46,56,62,78,84,99,114\} \\
N_{57} & = & \{8,12,50,53,57,64,71,73,90,99,110,119\} \\
N_{58} & = & \{9,12,22,38,42,44,51,58,65,66,68,76,88,91,97,103,107,108,124\} \\
N_{59} & = & \{9,13,21,34,59,91,116,119\} \\
N_{60} & = & \{7,8,24,30,37,40,60,123\} \\
N_{61} & = & \{1,12,16,27,40,47,61,86,99,110,114\} \\
N_{62} & = & \{6,20,62,64,70,82,87,91,98,115\} \\
N_{63} & = & \{5,14,32,34,37,54,63,82,91,93,101,109,114\} \\
N_{64} & = & \{3,6,9,23,53,64,85,96,100,109\} \\
N_{65} & = & \{2,4,5,17,23,26,38,51,65,80,92,103\} \\
N_{66} & = & \{28,37,38,46,66,72,90,99,100,104,113,122\} \\
N_{67} & = & \{16,18,38,67,120\} \\
N_{68} & = & \{40,52,68,71,95\} \\
N_{69} & = & \{5,42,52,58,69,70,95,106,108,116\} \\
N_{70} & = & \{26,59,70,76,78,114,116,119\} \\
N_{71} & = & \{11,28,31,44,51,54,58,64,71,77,96,112,115,120\} \\
N_{72} & = & \{2,10,22,23,44,72,113,116\} \\
N_{73} & = & \{15,25,73,78,115\} \\
N_{74} & = & \{9,20,27,30,37,54,60,68,74,79,97,107\} \\
N_{75} & = & \{23,27,32,34,43,48,59,65,75,89,99,112\} \\
N_{76} & = & \{4,10,25,28,39,48,62,76,118,125\} \\
N_{77} & = & \{1,20,47,51,56,69,75,77,84,87,89,97,122\} \\
N_{78} & = & \{2,75,78,89,91,124\} \\
N_{79} & = & \{7,8,20,47,54,79,91,116,121,124\} \\
N_{80} & = & \{7,28,30,39,59,64,71,72,75,80,91,93,102,117\} \\
N_{81} & = & \{2,15,53,54,63,76,79,81,91,117,119\} \\
N_{82} & = & \{7,9,24,25,33,71,79,82,100\} \\
N_{83} & = & \{3,4,24,29,31,36,41,83,86,102,107\} \\
N_{84} & = & \{12,21,22,37,55,84,111,122,125\} \\
N_{85} & = & \{23,29,48,50,85,95,97\} \\
N_{86} & = & \{42,52,80,86,96,97,99,107,110,112,116,121\} \\
N_{87} & = & \{5,28,38,52,56,57,65,87,109\} \\
N_{88} & = & \{21,22,24,30,37,50,55,75,86,88,120\} \\
N_{89} & = & \{4,13,22,28,29,49,62,89,90,91,118\} \\
N_{90} & = & \{12,17,20,26,30,54,58,76,90,92,93,98,104,120\} \\
N_{91} & = & \{24,29,33,41,49,51,91,111\} \\
N_{92} & = & \{5,22,27,38,45,53,54,59,69,73,92,109,110,112\} \\
N_{93} & = & \{6,13,34,59,69,78,93,109\} \\
N_{94} & = & \{19,22,35,37,45,54,55,58,68,70,89,94,96,113\} \\
N_{95} & = & \{2,7,50,78,95,118\} \\
N_{96} & = & \{32,65,66,76,90,93,96,112\} \\
N_{97} & = & \{15,33,62,88,95,97,100,104,106\} \\
N_{98} & = & \{13,37,51,74,86,95,98,100,112,115\} \\
N_{99} & = & \{6,39,43,99,121\} \\
N_{100} & = & \{17,21,22,34,40,41,61,70,90,100,108,112\} \\
N_{101} & = & \{3,8,43,57,96,101,103\} \\
N_{102} & = & \{8,30,34,42,47,52,65,77,85,102\} \\
N_{103} & = & \{59,63,65,72,94,95,103,108,124\} \\
N_{104} & = & \{4,15,21,57,77,82,96,104\} \\
N_{105} & = & \{7,28,46,47,56,69,75,82,85,93,97,105,112,118\} \\
N_{106} & = & \{27,32,38,106,107\} \\
N_{107} & = & \{7,23,29,33,61,63,107,113\} \\
N_{108} & = & \{17,23,33,47,60,63,65,75,78,91,92,101,108,111,122\} \\
N_{109} & = & \{28,77,80,84,102,104,109,111,124\} \\
N_{110} & = & \{8,24,27,28,47,64,94,100,102,110,116\} \\
N_{111} & = & \{12,18,23,53,61,65,74,93,108,109,111,118\} \\
N_{112} & = & \{16,18,19,34,47,50,52,66,71,104,112,114\} \\
N_{113} & = & \{51,55,63,80,100,113\} \\
N_{114} & = & \{2,73,89,107,114,125\} \\
N_{115} & = & \{8,10,16,23,32,41,47,61,63,70,76,79,89,93,115,118,120\} \\
N_{116} & = & \{21,30,40,43,46,82,105,116,123\} \\
N_{117} & = & \{25,38,85,100,101,117,124\} \\
N_{118} & = & \{25,47,69,79,93,118,120\} \\
N_{119} & = & \{5,10,22,27,34,56,66,67,71,85,87,91,98,108,117,119\} \\
N_{120} & = & \{4,31,48,49,52,55,59,73,91,93,120\} \\
N_{121} & = & \{10,27,44,95,98,121\} \\
N_{122} & = & \{10,14,35,49,63,75,91,97,102,122\} \\
N_{123} & = & \{4,17,20,28,52,73,93,96,98,111,117,120,123\} \\
N_{124} & = & \{40,52,73,77,78,96,101,117,119,124\} \\
N_{125} & = & \{3,10,34,43,92,105,108,125\}
\end{array} \right.$$

$$
\mathscr{A}_7 = \left\{ \begin{array}{lll}
N_1 & = & \{1,5,31,65,71,80,85,91,94,107\} \\
N_2 & = & \{2,8,11,19,27,31,38,47,48,50,58,76,89,98,102,104,110,116\} \\
N_3 & = & \{3,11,20,23,34,48,51,57,70,81,88,92,95,97,102,104\} \\
N_4 & = & \{2,3,4,22,39,59,67,71,95,124\} \\
N_5 & = & \{5,28,31,42,112,114\} \\
N_6 & = & \{6,27,35,37,48,82\} \\
N_7 & = & \{7,18,42,66,67,72,77,80,86,101,103,109,117,121\} \\
N_8 & = & \{8,11,21,28,32,57,74,86\} \\
N_9 & = & \{2,9,23,37,39,44,58,75,86,104,107,117,124\} \\
N_{10} & = & \{7,10,13,19,20,36,38,63,82,83,86,89,92,93,105,111,120,121\} \\
N_{11} & = & \{11,13,19,51,84,85,93,100\} \\
N_{12} & = & \{10,12,60,63,69,74,79,105,117\} \\
N_{13} & = & \{7,13,26,31,56,57,84,104,106,109\} \\
N_{14} & = & \{4,14,18,24,25,28,86,113\} \\
N_{15} & = & \{3,7,10,15,63,70,72,76,88,91,99,111,125\} \\
N_{16} & = & \{3,13,16,32,39,46,49,54,79,81,87,99,101,111,115,118\} \\
N_{17} & = & \{4,17,24,50,53,54,71,96,111,114\} \\
N_{18} & = & \{10,18,21,23,24,26,33,42,60,66,70,82,86,111,112,116\} \\
N_{19} & = & \{8,19,22,27,55,116\} \\
N_{20} & = & \{9,10,17,19,20,22,27,38,40,65,72,79,87,93,96\} \\
N_{21} & = & \{3,21,44,46,98,101\} \\
N_{22} & = & \{22,24,32,46,56,57,70,85,92\} \\
N_{23} & = & \{23,73,84,94,97,100,104,106,116,122\} \\
N_{24} & = & \{11,15,18,24,52,54,55,63,66,68,71,78,92\} \\
N_{25} & = & \{14,20,25,40,45,54,60,63,77,100,110\} \\
N_{26} & = & \{10,26,40,43,46,53,61,73,102,107,110,115,122,123\} \\
N_{27} & = & \{19,27,40,41,46,61,67,77,78,81,88,89,91,99,100,111,120,124\} \\
N_{28} & = & \{6,10,18,19,27,28,62,65,67,75,95,112\} \\
N_{29} & = & \{22,29,31,45,48,62,69,84,101\} \\
N_{30} & = & \{1,2,3,30,33,35,70,83,89,94,96,116\} \\
N_{31} & = & \{5,24,31,35,40,47,71,95,115,119\} \\
N_{32} & = & \{6,7,9,30,32,55,63,101,103,106,123\} \\
N_{33} & = & \{3,14,18,21,30,33,35,51,72,77,92,96,111\} \\
N_{34} & = & \{34,49,51,54,91,96,107,109,122\} \\
N_{35} & = & \{1,5,11,18,25,30,33,34,35,81,84,92\} \\
N_{36} & = & \{5,13,24,26,28,33,36,40,44,46,81,100,106,111,118,122\} \\
N_{37} & = & \{1,9,24,27,37,41,69,94,121\} \\
N_{38} & = & \{11,38,43,87,100,109\} \\
N_{39} & = & \{15,17,39,53,56,58,78,99,101,103\} \\
N_{40} & = & \{36,40,45,110\} \\
N_{41} & = & \{13,41,56,58,75,95\} \\
N_{42} & = & \{42,63,73,101,105,111,113\} \\
N_{43} & = & \{37,43,47,56\} \\
N_{44} & = & \{14,31,44,84,108\} \\
N_{45} & = & \{2,14,42,45,73,91,111\} \\
N_{46} & = & \{7,24,29,46,82,105,107\} \\
N_{47} & = & \{10,11,20,25,27,36,37,41,44,47,51,64,69,71,74,100,112,120\} \\
N_{48} & = & \{19,46,48,51,68,79,118\} \\
N_{49} & = & \{31,44,49,65,71,73,81,98\} \\
N_{50} & = & \{19,50,60,63,65,73,99,102\} \\
N_{51} & = & \{1,4,21,37,47,51,62,69,82,93,112,124\} \\
N_{52} & = & \{6,9,17,52,66,70,113,116\} \\
N_{53} & = & \{6,14,19,32,40,53,57,80,90,94,105,106,117,118,121,123\} \\
N_{54} & = & \{4,13,17,28,45,54,65,72,76,84,93,94,95,113,117,123\} \\
N_{55} & = & \{6,8,17,22,27,55,75,77,87,88,119\} \\
N_{56} & = & \{7,24,56,59,69,88,98,108\} \\
N_{57} & = & \{4,14,15,17,21,39,57,70,77,82,113\} \\
N_{58} & = & \{2,5,42,47,58,63,97,98,101,103,105,112,113\} \\
N_{59} & = & \{8,18,41,42,59,62,70,78,91,93,95,97,105,107,118\} \\
N_{60} & = & \{15,22,33,42,56,60,73,88,97,103,104,120\} \\
N_{61} & = & \{39,40,41,61,72,87,105,111,113\} \\
N_{62} & = & \{33,41,57,62,98,101,110,118,119\} \\
N_{63} & = & \{33,37,44,63,75,80,81,110\} \\
N_{64} & = & \{16,21,31,45,64,72,115,119\} \\
N_{65} & = & \{4,12,33,47,52,59,62,65,70,79,112\} \\
N_{66} & = & \{13,26,28,42,45,48,51,52,66,67,86,91,93,99,102,103\} \\
N_{67} & = & \{4,5,11,21,32,37,67,68,92,93,94,105,108,112,117\} \\
N_{68} & = & \{1,12,21,37,43,50,53,60,61,62,63,68,91,94,101,114,122\} \\
N_{69} & = & \{13,39,42,69,80,109\} \\
N_{70} & = & \{33,70,84,89,100,104,113\} \\
N_{71} & = & \{7,27,42,66,71,95,96,109\} \\
N_{72} & = & \{6,26,31,34,43,48,67,72,83,89,107,116\} \\
N_{73} & = & \{7,24,28,42,43,46,63,73,74,88,94,106,120\} \\
N_{74} & = & \{5,30,45,74,103,108,122,123,124,125\} \\
N_{75} & = & \{1,14,22,24,26,40,44,46,73,75,84,96,100\} \\
N_{76} & = & \{19,37,43,76,82,99,122\} \\
N_{77} & = & \{34,63,70,71,77,98,99,111\} \\
N_{78} & = & \{10,29,30,37,38,46,54,67,78,79,85,92,95,99,101,119\} \\
N_{79} & = & \{15,22,35,39,47,49,58,64,70,76,79,93,113\} \\
N_{80} & = & \{8,23,39,41,42,45,46,53,66,75,78,80\} \\
N_{81} & = & \{9,10,17,19,41,46,54,57,58,79,81,89,102,120\} \\
N_{82} & = & \{4,6,8,42,45,47,67,69,82,107\} \\
N_{83} & = & \{23,33,56,68,73,83,86,109,113,114\} \\
N_{84} & = & \{2,54,74,75,84,85,99,100,108,117\} \\
N_{85} & = & \{3,24,25,32,63,73,85,93,105,109,118\} \\
N_{86} & = & \{31,86,110,121\} \\
N_{87} & = & \{1,15,22,37,46,57,87,118\} \\
N_{88} & = & \{18,42,58,59,65,66,68,83,86,88,95,113,115,118,119\} \\
N_{89} & = & \{14,24,32,71,75,79,89,96,101,109,115,120\} \\
N_{90} & = & \{18,35,42,43,48,75,79,90,92,104,110\} \\
N_{91} & = & \{2,20,27,28,44,74,91,108\} \\
N_{92} & = & \{1,21,25,52,58,75,80,92,109,110\} \\
N_{93} & = & \{12,14,45,53,70,88,93,107,115,122\} \\
N_{94} & = & \{12,20,35,38,41,60,62,69,74,75,94,106,123,125\} \\
N_{95} & = & \{35,46,73,82,88,95,97,105\} \\
N_{96} & = & \{11,26,28,29,33,48,60,64,71,74,96,100,106,113,117\} \\
N_{97} & = & \{12,23,61,69,71,81,84,85,97,109,125\} \\
N_{98} & = & \{15,67,81,82,96,98,109,125\} \\
N_{99} & = & \{7,35,56,61,70,83,88,99,115,117\} \\
N_{100} & = & \{20,26,54,72,74,95,97,98,100,107,116,117\} \\
N_{101} & = & \{6,30,36,41,47,54,86,87,98,101,106,124\} \\
N_{102} & = & \{11,16,34,50,66,79,84,93,101,102,116\} \\
N_{103} & = & \{6,13,21,41,70,103,115,116\} \\
N_{104} & = & \{25,31,42,48,75,76,88,92,95,104\} \\
N_{105} & = & \{15,37,51,69,70,82,93,104,105,106,110\} \\
N_{106} & = & \{16,26,30,44,64,67,80,100,101,106\} \\
N_{107} & = & \{10,13,25,44,107,116\} \\
N_{108} & = & \{9,12,53,59,98,107,108,115\} \\
N_{109} & = & \{7,15,23,30,42,82,96,100,101,103,104,106,109,117\} \\
N_{110} & = & \{10,30,50,52,68,77,80,84,95,98,103,110\} \\
N_{111} & = & \{6,29,41,52,61,65,75,77,79,111,120\} \\
N_{112} & = & \{5,7,10,29,47,59,61,65,92,98,100,103,112\} \\
N_{113} & = & \{7,10,41,55,61,77,81,113\} \\
N_{114} & = & \{33,44,57,81,85,111,114,119,120,121\} \\
N_{115} & = & \{26,42,53,109,115,120\} \\
N_{116} & = & \{37,56,78,107,112,116,120,125\} \\
N_{117} & = & \{9,18,25,34,40,50,53,62,90,94,105,108,117,122\} \\
N_{118} & = & \{8,29,57,63,71,81,84,118\} \\
N_{119} & = & \{24,40,45,47,67,92,99,103,108,116,119\} \\
N_{120} & = & \{38,40,47,53,75,88,106,107,108,120\} \\
N_{121} & = & \{14,25,37,40,51,56,89,92,104,108,117,121\} \\
N_{122} & = & \{5,26,34,35,45,54,57,59,76,82,84,92,96,122,124\} \\
N_{123} & = & \{3,7,24,76,88,91,107,108,123\} \\
N_{124} & = & \{3,13,19,42,49,52,59,75,86,92,96,115,124\} \\
N_{125} & = & \{3,37,51,74,90,94,125\}
\end{array} \right.
$$

$$
\mathscr{A}_8 = \left\{ \begin{array}{lll}
N_1 & = & \{1,3,45,49,65,77,78,82,109,111\} \\
N_2 & = & \{2,13,28,31,32,36,44,76,87,97,106,107,113\} \\
N_3 & = & \{3,13,15,22,76,89,100,118\} \\
N_4 & = & \{4,18,19,34,37,47,57,58,66,71,76,125\} \\
N_5 & = & \{5,10,14,29,43,78,95,113\} \\
N_6 & = & \{6,9,31,37,38,61,70,87,100,103,104,112,124,125\} \\
N_7 & = & \{7,8,9,12,29,37,40,69,123\} \\
N_8 & = & \{5,8,41,61,66,70,80,83,100,116\} \\
N_9 & = & \{7,9,22,23,37,83,85,90,93,103,106\} \\
N_{10} & = & \{10,63,77,106,122,123\} \\
N_{11} & = & \{4,10,11,19,25,33,53,70,88,109,110\} \\
N_{12} & = & \{9,12,50,72,83,93,119\} \\
N_{13} & = & \{9,13,17,55,61,76,78,98,99,106\} \\
N_{14} & = & \{3,4,14,71,80,89,110\} \\
N_{15} & = & \{1,15,21,23,65,75,125\} \\
N_{16} & = & \{16,28,34,84,101,118,122\} \\
N_{17} & = & \{1,4,17,32,38,101,124\} \\
N_{18} & = & \{1,17,18,25,37,48,63,73,96,106,116\} \\
N_{19} & = & \{19,43,66,98,102,119\} \\
N_{20} & = & \{20,39,43,50,63,66,73,81,84,89,90,100,107,108,116,117\} \\
N_{21} & = & \{21,32,60,75,84\} \\
N_{22} & = & \{14,19,22,26,38,41,43,46,48,53,70,103\} \\
N_{23} & = & \{3,5,23,28,35,37,58,66,73,76,95,102\} \\
N_{24} & = & \{8,24,25,28,39,46,47,58,77,98,106\} \\
N_{25} & = & \{25,34,56,88,92,98,109,122\} \\
N_{26} & = & \{16,26,51,65,66,100,101,109,121\} \\
N_{27} & = & \{14,16,27,42,52,70,111,117,118,120,121,125\} \\
N_{28} & = & \{2,12,18,28,36,40,61,94,102,125\} \\
N_{29} & = & \{29,37,44,68,77,78,97,99\} \\
N_{30} & = & \{1,27,30,40,55,95,100,115,124\} \\
N_{31} & = & \{19,28,31,37,53,67,80,82,83,92,102,103,116,117\} \\
N_{32} & = & \{18,27,32,45,48,57,70,81,83\} \\
N_{33} & = & \{12,27,33,44,61,66,76,85\} \\
N_{34} & = & \{11,12,21,34,53,60,76,84,92,99,108,125\} \\
N_{35} & = & \{2,26,32,35,37,49,64,66,78,86,95,100,103,108\} \\
N_{36} & = & \{8,10,21,26,29,36,40,99,108,111,121,123\} \\
N_{37} & = & \{12,20,22,37,70,89,100\} \\
N_{38} & = & \{38,56,79\} \\
N_{39} & = & \{19,39,67,92,98\} \\
N_{40} & = & \{1,9,12,15,27,36,40,42,47,57,68,86,104,106\} \\
N_{41} & = & \{9,19,30,41,99\} \\
N_{42} & = & \{2,42,43,46,54,65,72,89,95,109,113,117,124\} \\
N_{43} & = & \{1,4,7,17,37,43,49,51,95,97,114\} \\
N_{44} & = & \{28,44,57,78,83,84,87,88,99,108\} \\
N_{45} & = & \{12,19,24,34,45,53,58,74,75,80,113\} \\
N_{46} & = & \{3,10,14,23,32,40,46,56,75,89,119\} \\
N_{47} & = & \{27,29,34,47,76,83,96,105,120\} \\
N_{48} & = & \{11,12,21,25,48,81,88,89,99,124\} \\
N_{49} & = & \{47,49,91,113,118\} \\
N_{50} & = & \{4,26,50,78,82,87,101,113,123\} \\
N_{51} & = & \{6,7,26,36,51,73,93,117\} \\
N_{52} & = & \{6,11,19,24,31,37,43,52,58,72,74,104,107\} \\
N_{53} & = & \{11,12,14,20,21,27,53,54,55,62\} \\
N_{54} & = & \{2,45,54,69,77,86,94,95\} \\
N_{55} & = & \{16,37,48,54,55,56,59,77,83,89,96,102,108,117\} \\
N_{56} & = & \{3,18,28,31,33,42,56,58,84,98,103,105,107,108\} \\
N_{57} & = & \{19,25,35,54,57,60,78,85,86,112,122\} \\
N_{58} & = & \{7,44,58,67,68,107\} \\
N_{59} & = & \{11,12,42,52,59,87,88,94,110,112\} \\
N_{60} & = & \{47,50,60,66,81,114,121\} \\
N_{61} & = & \{1,53,56,61,78,88,89,102,104,111,112\} \\
N_{62} & = & \{17,21,51,62,82,85,102,115,116,124\} \\
N_{63} & = & \{27,48,51,58,59,63,79,101,102\} \\
N_{64} & = & \{38,43,49,57,64,70,75,79,81,88,101,113,114,115,116\}
\end{array} \right.
$$

$$\mathscr{A}_8 = \left\{ \begin{array}{lll}
N_{65} & = & \{10,20,31,35,48,53,59,65,89,98,109,112,116,119,124\} \\
N_{66} & = & \{42,53,58,65,66,79,90,98,100,119\} \\
N_{67} & = & \{13,25,26,34,52,60,67,102,124\} \\
N_{68} & = & \{8,68,69,70,74,80,82,102,114\} \\
N_{69} & = & \{6,8,13,69,103,113\} \\
N_{70} & = & \{7,11,14,18,25,37,52,70,71,76,77,110\} \\
N_{71} & = & \{9,22,23,44,47,71,96,110\} \\
N_{72} & = & \{11,17,22,23,24,41,44,46,51,54,72,99,107,109\} \\
N_{73} & = & \{5,25,37,46,50,52,73,78,120\} \\
N_{74} & = & \{8,10,16,22,54,57,63,70,71,74,91,94\} \\
N_{75} & = & \{8,13,16,23,57,60,75,112,121\} \\
N_{76} & = & \{8,22,25,29,30,33,55,76,95,99,111,123\} \\
N_{77} & = & \{9,22,51,65,68,73,77,80,97,102,108,109\} \\
N_{78} & = & \{11,25,35,53,71,78,86,90,104,111,117,120\} \\
N_{79} & = & \{34,35,43,78,79,90,103,110\} \\
N_{80} & = & \{9,47,55,64,66,75,78,80,82,120\} \\
N_{81} & = & \{3,15,59,81,86,100,106,116\} \\
N_{82} & = & \{12,14,21,33,36,43,82,89,109\} \\
N_{83} & = & \{6,39,48,73,79,83,88,89,95,102,114\} \\
N_{84} & = & \{11,12,20,42,68,84,89,99\} \\
N_{85} & = & \{6,7,17,23,45,46,54,60,65,80,82,85,99,112,119,123\} \\
N_{86} & = & \{4,24,32,49,51,53,55,85,86,93,99,101,107,117,121\} \\
N_{87} & = & \{14,17,21,28,36,81,87,90,96,102,103,120\} \\
N_{88} & = & \{3,86,88,89,99,111\} \\
N_{89} & = & \{6,18,29,37,39,41,46,70,89,91,99\} \\
N_{90} & = & \{7,34,42,53,71,90,104,108,115\} \\
N_{91} & = & \{30,54,91,105,108,118\} \\
N_{92} & = & \{9,14,21,51,64,79,89,92,96,125\} \\
N_{93} & = & \{12,26,31,40,62,67,93,123\} \\
N_{94} & = & \{17,35,76,91,94,103,104,106,121\} \\
N_{95} & = & \{14,22,26,40,63,74,95\} \\
N_{96} & = & \{46,58,90,96,114,122\} \\
N_{97} & = & \{4,11,12,39,40,55,90,94,97,117,119\} \\
N_{98} & = & \{1,3,21,26,36,52,76,98,112\} \\
N_{99} & = & \{3,16,21,26,42,99,104,115,125\} \\
N_{100} & = & \{29,33,45,52,61,68,71,100,117\} \\
N_{101} & = & \{2,3,9,17,20,26,54,61,72,75,78,98,101,102,125\} \\
N_{102} & = & \{2,7,9,33,39,48,66,100,102\} \\
N_{103} & = & \{18,27,31,32,47,53,59,75,88,93,103,109\} \\
N_{104} & = & \{48,100,104,111,119,120\} \\
N_{105} & = & \{7,17,26,36,37,46,103,105,111\} \\
N_{106} & = & \{8,13,31,50,52,67,98,106,108,114\} \\
N_{107} & = & \{32,56,66,106,107,113\} \\
N_{108} & = & \{25,56,94,108\} \\
N_{109} & = & \{1,5,10,28,50,54,60,61,77,85,97,101,105,109,117\} \\
N_{110} & = & \{28,35,42,52,64,79,83,104,110\} \\
N_{111} & = & \{15,51,68,86,92,104,111,117,121,122\} \\
N_{112} & = & \{10,36,63,64,98,105,112\} \\
N_{113} & = & \{10,79,88,95,113,120\} \\
N_{114} & = & \{1,8,10,12,37,44,86,114,116,118\} \\
N_{115} & = & \{12,29,34,42,43,47,51,54,57,62,68,69,92,97,100,105,110,115\} \\
N_{116} & = & \{2,17,43,64,73,83,102,116,120\} \\
N_{117} & = & \{1,4,9,12,28,30,33,43,74,116,117\} \\
N_{118} & = & \{11,12,15,22,53,55,72,81,88,96,117,118\} \\
N_{119} & = & \{3,29,60,78,82,85,119\} \\
N_{120} & = & \{31,39,57,75,89,91,95,99,101,103,106,120,123\} \\
N_{121} & = & \{3,5,8,14,26,35,49,56,61,71,77,100,109,114,116,121\} \\
N_{122} & = & \{3,13,20,28,37,46,47,48,51,52,60,61,72,73,112,122\} \\
N_{123} & = & \{1,14,32,33,60,64,81,101,121,123\} \\
N_{124} & = & \{24,25,71,90,124\} \\
N_{125} & = & \{2,7,13,22,28,36,47,59,102,104,118,125\}
\end{array} \right.$$

$$\mathscr{A}_9 = \left\{ \begin{array}{lll}
N_{1} & = & \{1,4,21,48,57,86,110,112,114,115\} \\
N_{2} & = & \{2,7,11,12,24,25,28,37,63,91,93,107,116,125\} \\
N_{3} & = & \{2,3,11,36,39,55,66,90,98,113\} \\
N_{4} & = & \{4,49,65,85,123\} \\
N_{5} & = & \{5,11,15,17,24,56,57,66,76,81,99,102\} \\
N_{6} & = & \{1,6,46,74,100,101,105,120,125\} \\
N_{7} & = & \{7,32,41,54,59,66,76,77,83,84,86,101,112,121\} \\
N_{8} & = & \{8,32,54,57,58,59,65,75,109,120,124\} \\
N_{9} & = & \{7,9,43,50,66,84,87\} \\
N_{10} & = & \{10,25,29,59,62,81,92,110,118\} \\
N_{11} & = & \{8,9,11,34,54,57,72,85,98,102\} \\
N_{12} & = & \{12,32,42,62,64,76,105\} \\
N_{13} & = & \{4,13,20,81,112\} \\
N_{14} & = & \{9,12,14,61,73,75,89,95,99\} \\
N_{15} & = & \{8,15,36,43,44,82,84,98,105,106,113,119,123\} \\
N_{16} & = & \{14,16,23,33,82,85,107,116\} \\
N_{17} & = & \{17,33,47,60,65,74,82,92,101\} \\
N_{18} & = & \{18,21,23,29,62,64,78,92,107,108,117\} \\
N_{19} & = & \{1,10,19,40,45,56,67,76,90,91,109,119\} \\
N_{20} & = & \{1,4,20,29,48,60,64,67,72,99,110,123\} \\
N_{21} & = & \{17,21,66,78,83,90,120\} \\
N_{22} & = & \{10,11,13,20,22,35,37,56,69,90,110\} \\
N_{23} & = & \{23,36,41,73,84\} \\
N_{24} & = & \{13,24,32,52,61,70,96,98,112,121\} \\
N_{25} & = & \{15,25,67,74,85,87,117\} \\
N_{26} & = & \{8,14,19,20,22,24,26,49,51,61,64,89,95,108,113,115,116,124\} \\
N_{27} & = & \{1,10,17,22,25,27,56,73,77,85,97,109,122,124\} \\
N_{28} & = & \{1,28,42,45,46,65,88,107,108,110,115,121,122\} \\
N_{29} & = & \{3,25,29,45,64,118,121,123,124\} \\
N_{30} & = & \{7,11,13,28,30,71,77,102,107,109,115,125\} \\
N_{31} & = & \{8,31,53,79,89,90,91,98,120\} \\
N_{32} & = & \{14,32,65,88,100,104,112\} \\
N_{33} & = & \{18,30,33,39,43,71,84,89,98,105\} \\
N_{34} & = & \{5,9,16,26,27,34,38,50,105,110,114\} \\
N_{35} & = & \{2,10,15,24,35,41,48,55,60,82,84,104,107,108\} \\
N_{36} & = & \{4,5,8,26,27,36,56,64,84,88,95,104,113,116\} \\
N_{37} & = & \{6,29,37,46,62,99,100,105,123\} \\
N_{38} & = & \{11,38,82,109\} \\
N_{39} & = & \{20,29,35,39,49,61,71,94,102,116,121\} \\
N_{40} & = & \{19,40,50,53,68,71,75,77,78,81,92,101\} \\
N_{41} & = & \{18,31,41,56,57,77,83,94,101,104,109,119\} \\
N_{42} & = & \{5,6,33,42,66,94,103,106,124\} \\
N_{43} & = & \{28,43,47,85,86,118\} \\
N_{44} & = & \{2,13,35,44,45,89,115\} \\
N_{45} & = & \{25,28,33,44,45,53,57,62,64,67,71,81,82,101,108,114\} \\
N_{46} & = & \{4,19,21,46,47,57,60,89,90,97,115,120\} \\
N_{47} & = & \{25,38,47,59,62,74,124\} \\
N_{48} & = & \{35,41,48,55,58,59,74,95\} \\
N_{49} & = & \{2,3,7,14,17,27,48,49,69,95,96,111,118\} \\
N_{50} & = & \{39,43,50,53,70,101,107,108,124\} \\
N_{51} & = & \{19,22,24,51,62,63,68,69,78,117,120\} \\
N_{52} & = & \{28,52,61,64,68,76,91,96\} \\
N_{53} & = & \{13,14,19,26,32,48,53,55,84\} \\
N_{54} & = & \{9,23,45,50,54,59,60,65,76,81,93\} \\
N_{55} & = & \{6,10,17,34,35,42,55,68,87\} \\
N_{56} & = & \{56,58,81,105,111,112\} \\
N_{57} & = & \{18,31,45,48,57,90,100,104,113,120,122\} \\
N_{58} & = & \{5,16,58,94,102,105,110\} \\
N_{59} & = & \{5,25,48,59,62,76,83,113,114\} \\
N_{60} & = & \{37,52,60,78,87,96,120,125\} \\
N_{61} & = & \{22,29,54,57,61,65,78,89\} \\
N_{62} & = & \{13,17,31,34,48,62,79,95,116,117,120\} \\
N_{63} & = & \{13,30,52,63,77,79,112,124\} \\
N_{64} & = & \{5,7,12,24,31,32,44,58,60,64,72,74,81,95,100\} \\
N_{65} & = & \{9,31,32,65,77,85,87,89,98,99,108\} \\
N_{66} & = & \{29,66,75,79,88,91,97,110\} \\
N_{67} & = & \{12,55,67,76,106\} \\
N_{68} & = & \{8,22,36,43,68,70,75,81,86,87,110,113,116\} \\
N_{69} & = & \{5,39,46,55,61,69,82,86,91,104,116\} \\
N_{70} & = & \{10,14,48,54,68,70,85,90\} \\
N_{71} & = & \{5,16,38,44,46,47,71,77,108,111\} \\
N_{72} & = & \{3,20,21,27,72,88,96,99,102,117\} \\
N_{73} & = & \{4,37,47,56,70,73,75,97,109,112\} \\
N_{74} & = & \{31,48,56,60,72,73,74,113\} \\
N_{75} & = & \{15,18,22,45,47,56,69,75,78,81,115,123\} \\
N_{76} & = & \{1,10,33,40,41,59,68,74,75,76,78,79\} \\
N_{77} & = & \{2,5,13,20,33,34,77,98,104,114,116,118,120\} \\
N_{78} & = & \{47,59,78,80,106\} \\
N_{79} & = & \{15,22,27,28,38,79,95,96,102\} \\
N_{80} & = & \{6,10,25,49,69,76,80,98,117,121,124\} \\
N_{81} & = & \{2,9,10,11,13,28,30,41,44,57,65,67,75,81,89,107,111\} \\
N_{82} & = & \{7,13,25,37,41,47,55,81,82,99,101\} \\
N_{83} & = & \{24,53,68,69,72,83,94,117\} \\
N_{84} & = & \{23,31,61,66,79,84,96,104,111,115\} \\
N_{85} & = & \{6,47,51,56,61,85,94,101,106,118,124\} \\
N_{86} & = & \{19,20,44,55,65,72,75,78,86,93,120\} \\
N_{87} & = & \{8,10,14,20,72,85,87,104,112\} \\
N_{88} & = & \{11,18,39,40,48,88,90,106,108\} \\
N_{89} & = & \{25,34,38,49,84,89,100,107,123\} \\
N_{90} & = & \{19,32,35,48,49,60,63,71,81,90,105,117\} \\
N_{91} & = & \{4,15,21,23,59,65,75,88,91,93,101,124\} \\
N_{92} & = & \{31,45,56,63,69,73,76,86,92,118,124\} \\
N_{93} & = & \{51,56,78,89,93,103,109\} \\
N_{94} & = & \{1,4,6,14,32,64,94,98\} \\
N_{95} & = & \{11,21,36,42,50,81,90,95,113,117\} \\
N_{96} & = & \{4,17,32,52,78,83,93,95,96,115,119,121\} \\
N_{97} & = & \{17,22,24,30,40,68,73,82,84,97,113\} \\
N_{98} & = & \{9,18,50,61,67,83,91,98,114,122\} \\
N_{99} & = & \{14,36,45,71,76,99,112,124\} \\
N_{100} & = & \{6,12,22,58,77,89,92,98,99,100,106,107,109,124\} \\
N_{101} & = & \{5,36,48,55,72,82,86,101\} \\
N_{102} & = & \{11,12,32,50,64,68,102,117,125\} \\
N_{103} & = & \{9,12,63,96,98,103,123\} \\
N_{104} & = & \{39,63,76,79,82,103,104,108,115,122\} \\
N_{105} & = & \{9,17,26,66,81,82,91,105,114\} \\
N_{106} & = & \{27,28,37,71,82,106,107,125\} \\
N_{107} & = & \{2,14,22,48,59,102,107\} \\
N_{108} & = & \{24,47,48,60,64,68,76,77,87,90,100,108,113\} \\
N_{109} & = & \{13,19,40,45,64,66,87,101,109,114\} \\
N_{110} & = & \{17,34,40,108,110\} \\
N_{111} & = & \{8,10,12,20,24,35,38,39,86,111\} \\
N_{112} & = & \{4,31,36,43,46,60,61,79,103,112\} \\
N_{113} & = & \{32,37,39,58,60,63,71,77,87,96,99,103,105,108,111,113\} \\
N_{114} & = & \{12,30,76,78,82,84,94,114,124\} \\
N_{115} & = & \{11,41,57,75,76,115,119\} \\
N_{116} & = & \{6,8,46,51,68,85,103,116,117\} \\
N_{117} & = & \{1,11,25,31,49,56,60,85,111,114,117,121\} \\
N_{118} & = & \{16,29,63,83,89,90,101,110,118\} \\
N_{119} & = & \{32,53,74,86,108,119\} \\
N_{120} & = & \{14,17,47,56,63,86,90,115,120,123,124\} \\
N_{121} & = & \{2,14,21,34,76,80,88,93,97,118,121\} \\
N_{122} & = & \{5,14,23,35,56,70,77,92,114,115,122,124\} \\
N_{123} & = & \{36,49,53,77,87,97,110,123\} \\
N_{124} & = & \{1,8,14,19,29,41,61,62,77,88,106,115,124\} \\
N_{125} & = & \{19,27,44,64,73,79,80,85,106,108,122,125\}
\end{array} \right.$$

$$
\mathscr{A}_{10} = \left\{
\begin{array}{lcl}
N_{1} & = & \{1,22,29,48,56,57,63,72,87,103,108,109,116,118,124\} \\
N_{2} & = & \{1,2,6,7,13,38,63,77,94,106\} \\
N_{3} & = & \{3,12,21,44,45,69,73,78,83,84,86,113,120,124\} \\
N_{4} & = & \{4,23,59,60\} \\
N_{5} & = & \{5,15,18,19,25,30,35,39,47,57,59,63,67,76,116\} \\
N_{6} & = & \{6,17,35,37,39,47,82,108,112,118,123\} \\
N_{7} & = & \{7,23,29,38,40,60,67,69,77,78,82,119\} \\
N_{8} & = & \{8,19,22,54,82,98,115,120\} \\
N_{9} & = & \{9,28,39,43,55,78,79,80,87,89,92,101,102,109,113,118,125\} \\
N_{10} & = & \{2,10,17,23,51,89,97,114,116,124\} \\
N_{11} & = & \{11,15,59,60,65,70,102,110,118,122\} \\
N_{12} & = & \{12,30,48,56,70,79,85,92,100\} \\
N_{13} & = & \{6,13,18,26,27,31,47,49,53,69,79,85,86,87,94,98,103,111,125\} \\
N_{14} & = & \{12,14,19,71,84,90,92,102,110,122\} \\
N_{15} & = & \{2,8,10,13,15,21,24,66,69,81,90,118,120\} \\
N_{16} & = & \{3,9,16,33,38,46,49,56,86,101,103,107,108\} \\
N_{17} & = & \{10,17,20,61,71,88,111,113,116,125\} \\
N_{18} & = & \{2,10,18,42,43,56,67,91,103,120,125\} \\
N_{19} & = & \{2,3,19,30,43,63,64,93,109,125\} \\
N_{20} & = & \{11,13,20,36,121\} \\
N_{21} & = & \{7,21,50,54,121\} \\
N_{22} & = & \{21,22,26,39,40,46,60,68,74,80,89,107\} \\
N_{23} & = & \{23,31,41,51,59,82,87,105,110,112,119,123\} \\
N_{24} & = & \{14,18,21,24,37,49,51,52,61,73,78,103\} \\
N_{25} & = & \{12,15,18,25,39,41,46,55,116\} \\
N_{26} & = & \{4,19,20,26,39,48,52,80,88,91,96,105,106,113,119,124\} \\
N_{27} & = & \{4,26,27,42,65,79,84,102,114,120\} \\
N_{28} & = & \{23,24,27,28,30,51,76,103,104,114,121\} \\
N_{29} & = & \{10,18,21,25,27,29,52,60,72,76,91,98,107\} \\
N_{30} & = & \{4,5,9,14,24,30,38,40,50,61,78,98,112\} \\
N_{31} & = & \{22,31,33,36,61,62,65,67,100,107,108,123\} \\
N_{32} & = & \{19,32,36,37,45,51,87,88,112\} \\
N_{33} & = & \{7,10,20,23,25,33,45,48,76,84,89,103,111,115,118,120\} \\
N_{34} & = & \{31,34,47,55,62,91,104\} \\
N_{35} & = & \{4,10,14,15,16,22,35,54,74,84,108\} \\
N_{36} & = & \{25,28,33,34,36,51,71,73,77,94,101,109,110\} \\
N_{37} & = & \{12,31,37,45,48,61,67,70,87\} \\
N_{38} & = & \{38,43,64,119\} \\
N_{39} & = & \{23,30,39,46,48,57,60,62,64,65,87,97,111,118\} \\
N_{40} & = & \{1,33,40,52,59,68,85,87,100\} \\
N_{41} & = & \{11,16,28,30,33,36,41,48,97,104,109,111,122\} \\
N_{42} & = & \{3,25,26,28,42,47,61,64,91\} \\
N_{43} & = & \{1,8,23,34,38,43,49,51,66,76,81,89,92,94,112,116,117,124\} \\
N_{44} & = & \{13,41,44,57,58,65,70,77,90\} \\
N_{45} & = & \{3,45,70,78,117\} \\
N_{46} & = & \{13,24,29,46,56,78,86,94,99,109,111,116,118\} \\
N_{47} & = & \{4,5,13,27,30,36,47,55,65,66,71,85,113\} \\
N_{48} & = & \{1,7,21,22,23,48,50,80,89,100,120,125\} \\
N_{49} & = & \{12,13,26,49,56,90,91,112,117,122\} \\
N_{50} & = & \{6,16,38,50,53,60,75,94,107,118,124\} \\
N_{51} & = & \{32,49,51,54,57,66,67,75,94,101,108\} \\
N_{52} & = & \{52,61,62,72,94\} \\
N_{53} & = & \{3,4,16,27,37,53,54,62,64,88,93,95,107\} \\
N_{54} & = & \{6,25,46,54,58,77,82,97,111\} \\
N_{55} & = & \{26,43,49,55,84,86,101,114,120,125\} \\
N_{56} & = & \{20,55,56,58,71,119\} \\
N_{57} & = & \{56,57,93,106\} \\
N_{58} & = & \{4,5,9,19,29,30,31,40,52,58,66,71,77,81,88,91,104,108,120,124\} \\
N_{59} & = & \{2,3,15,17,40,42,55,59,67,74,81,92,104,108,116\} \\
N_{60} & = & \{3,5,28,44,46,51,60,71,93,99,108\} \\
N_{61} & = & \{45,46,61,83,103,110\} \\
N_{62} & = & \{19,24,44,62,67,71,96,97,114\} \\
N_{63} & = & \{3,26,29,34,40,49,63,89\} \\
N_{64} & = & \{27,53,64,67\} \\
N_{65} & = & \{12,44,57,65,97,98,105,108,109,118\} \\
N_{66} & = & \{9,33,45,54,66,82,100,104\} \\
N_{67} & = & \{15,21,32,36,43,67,74,78,93,106,124\} \\
N_{68} & = & \{26,27,44,48,68,70,79,81,95,97,108,116,119\} \\
N_{69} & = & \{32,50,51,56,69,80,95,96,119,124\} \\
N_{70} & = & \{13,24,39,49,52,54,59,64,70,74,103,114\} \\
N_{71} & = & \{4,28,33,35,37,58,64,70,71,81,91\} \\
N_{72} & = & \{4,20,65,72,88,91,120,121,125\} \\
N_{73} & = & \{8,27,30,35,62,73,95,102,117\} \\
N_{74} & = & \{9,10,42,67,74,101,114\} \\
N_{75} & = & \{24,25,30,38,48,55,75,77,82,86,90,120\} \\
N_{76} & = & \{3,7,10,17,22,24,28,42,53,57,76,101\} \\
N_{77} & = & \{23,51,57,72,77,81,82,95,98,106,112,122\} \\
N_{78} & = & \{13,19,41,46,78,82,90,91,113\} \\
N_{79} & = & \{7,27,34,44,47,59,64,79,95,101\} \\
N_{80} & = & \{7,19,80,84,85,91,104\} \\
N_{81} & = & \{25,34,40,71,81,89,93,101,111\} \\
N_{82} & = & \{26,28,34,38,80,82,87,101\} \\
N_{83} & = & \{16,39,41,61,73,83,96,103,107,111\} \\
N_{84} & = & \{49,64,78,84,90,93,96,99,106,113\} \\
N_{85} & = & \{2,12,17,26,32,44,50,51,61,66,85,104,105,107,108,120\} \\
N_{86} & = & \{9,44,51,86,93,105,110,111,116,125\} \\
N_{87} & = & \{8,22,47,58,62,87,104,111\} \\
N_{88} & = & \{3,20,47,58,65,70,76,88,114,123\} \\
N_{89} & = & \{17,25,32,34,45,48,58,64,65,71,84,86,88,89,103\} \\
N_{90} & = & \{10,26,32,38,60,73,77,78,86,90,92,124\} \\
N_{91} & = & \{6,25,32,42,78,91,100,117\} \\
N_{92} & = & \{10,24,37,38,43,89,92,105\} \\
N_{93} & = & \{8,13,18,32,37,47,81,85,92,93,99,125\} \\
N_{94} & = & \{2,4,33,36,38,48,51,52,61,83,94,97,104,106,109,120\} \\
N_{95} & = & \{18,19,21,29,43,52,58,62,67,73,82,95,104,119\} \\
N_{96} & = & \{17,47,71,76,77,96,118,119,122\} \\
N_{97} & = & \{3,7,12,50,51,54,79,93,97,101,104\} \\
N_{98} & = & \{3,33,51,61,67,80,98,113\} \\
N_{99} & = & \{8,31,47,53,57,86,99,119,121\} \\
N_{100} & = & \{4,10,22,31,59,70,73,97,100\} \\
N_{101} & = & \{18,25,28,34,40,61,101,102,114\} \\
N_{102} & = & \{8,21,38,47,49,68,70,75,96,102,105,108\} \\
N_{103} & = & \{4,8,14,19,21,30,47,62,66,76,79,90,102,103,125\} \\
N_{104} & = & \{3,9,18,22,37,44,45,46,47,53,58,59,66,68,95,104,125\} \\
N_{105} & = & \{9,13,44,61,62,63,73,89,105\} \\
N_{106} & = & \{4,14,23,44,54,72,83,97,106,109,121\} \\
N_{107} & = & \{24,37,39,76,81,85,104,107,122\} \\
N_{108} & = & \{12,26,35,36,38,39,69,78,82,96,97,107,108,109\} \\
N_{109} & = & \{2,17,35,36,40,52,58,66,75,87,89,99,109,123\} \\
N_{110} & = & \{2,30,39,41,58,95,110,118\} \\
N_{111} & = & \{6,58,70,87,96,99,103,111,117\} \\
N_{112} & = & \{2,28,33,39,40,42,68,95,104,110,112\} \\
N_{113} & = & \{3,7,17,21,39,52,63,89,93,98,102,113,115,118\} \\
N_{114} & = & \{13,22,45,55,58,70,75,111,114,117\} \\
N_{115} & = & \{18,20,44,95,115,122\} \\
N_{116} & = & \{20,23,32,50,70,76,84,85,102,104,116\} \\
N_{117} & = & \{5,13,20,41,53,70,82,84,110,117,121\} \\
N_{118} & = & \{11,49,70,87,89,97,99,102,112,118\} \\
N_{119} & = & \{28,64,82,83,88,90,91,93,105,119\} \\
N_{120} & = & \{4,8,21,27,79,80,90,93,120,122\} \\
N_{121} & = & \{7,8,72,84,99,102,121\} \\
N_{122} & = & \{7,20,33,37,60,85,101,105,107,120,122\} \\
N_{123} & = & \{15,54,59,65,89,92,98,103,115,122,123\} \\
N_{124} & = & \{3,6,12,21,28,40,44,62,64,86,93,95,97,121,124,125\} \\
N_{125} & = & \{30,45,69,110,122,125\}
\end{array}
\right.
$$

As in the first example, we consider the evolution matrices F_j, associated with $\mathscr{A}_j$ for $j = 1, \ldots, 10$. The set $\mathscr{F} = \{F_j\}_{j=1}^{10}$ has now a mean density around 0.074. Applying the Gripenberg algorithm using the spectral norm, we obtain

$$\overline{\rho}_{20}(\mathscr{F}) = 0.996161 \leq \rho_2(\mathscr{F}) \leq 0.998234 = \widehat{\rho}_{20}(\mathscr{F}).$$

This result allows to conclude that the system will reach always a global consensus, i.e. the solution of equation (3.12) tends to be, for $t \to \infty$, a vector with equal entries.

As in the first example, also this set has an evolution matrix, F_4, whose corresponding topology allow to identify a leader/sink among the agents.

The index of this particular agent is 89. The matrix F_4, which has a second spectral radius almost equal 1, proves to be the candidate s.m.p. of the set $\mathscr{F}$. Its PageRank is a vector whose entries are all zero except the one in position 89. Therefore, if the system of agents maintain for every time step the topology associated to F_4, the solution $\theta(t)$ of (3.12), which can be represented by

$$\theta(t) = F_{i_{t-1}} \cdot \ldots \cdot F_{i_1} \cdot F_{i_0} \theta(0) \tag{B.3}$$

where $i_k = 4$, for every $k = 0, \ldots, t-1$, converges to the vector of all equal entries $e\theta_{cons}$ with $\theta_{cons} = \theta_{89}(0)$. The convergence is slow due to the second spectral radius of F_4.

If, instead, we allow the sequence of indices $\bar{i} = (i_0, i_1, \ldots)$ to assume at least once a value different from 4, we witness a change in the value θ_{cons} which is, in general, radical. This phenomenon can be interpreted as a *butterfly effect*: small changes either in the initial configuration $\theta(0)$ or in the chosen sequence of evolution matrices, can determine substantial differences in the final configuration of the system.

As an example we consider the following vector $\theta(0)$ which has mean value equal to 0.481868 and $\|\theta(0)\|_\infty = 1$.

$$\begin{aligned}
\theta(0)^T = (&0.79829\ ,\ 0.18760,\ 0.49166,\ 0.44732,\ 0.64882,\ 0.71212,\ 0.75762,\ 0.27710,\ 0.68234\ \ 0.65764,\\
&0.16324,\ 0.11946,\ 0.50030,\ 0.96347,\ 0.34171,\ 0.58754,\ 0.22468,\ 0.75418,\ 0.25608,\ 0.50792,\\
&0.70179,\ 0.89436,\ 0.96301,\ 0.54934,\ 0.13916,\ 0.14987,\ 0.25851,\ 0.84398,\ 0.25527,\ 0.81744,\\
&0.24447,\ 0.93287,\ 0.35134,\ 0.19736,\ 0.25206,\ 0.61844,\ 0.47513,\ 0.35302,\ 0.83405,\ 0.58754,\\
&0.55186,\ 0.92075,\ 0.28695,\ 0.76014,\ 0.75665,\ 0.38192,\ 0.57002,\ 0.07615,\ 0.05416,\ 0.53286,\\
&0.78219,\ 0.93763,\ 0.13041,\ 0.57103,\ 0.47121,\ 0.01195,\ 0.33843,\ 0.16281,\ 0.79737,\ 0.31242,\\
&0.53058,\ 0.16629,\ 0.60432,\ 0.26399,\ 0.65662,\ 0.69189,\ 0.75105,\ 0.45229,\ 0.08415,\ 0.22987,\\
&0.91688,\ 0.15297,\ 0.82902,\ 0.54043,\ 1.00000,\ 0.07848,\ 0.44440,\ 0.10707,\ 0.96563,\ 0.00465,\\
&0.77792,\ 0.82047,\ 0.87207,\ 0.08476,\ 0.40133,\ 0.26088,\ 0.80317,\ 0.43309,\ 0.91418,\ 0.18255,\\
&0.26483,\ 0.14610,\ 0.13660,\ 0.87267,\ 0.58195,\ 0.55199,\ 0.14552,\ 0.85634,\ 0.62447,\ 0.35231,\\
&0.51524,\ 0.40337,\ 0.07626,\ 0.24085,\ 0.12380,\ 0.18462,\ 0.24088,\ 0.41889,\ 0.04985,\ 0.90622,\\
&0.94845,\ 0.49277,\ 0.49115,\ 0.33903,\ 0.90355,\ 0.37068,\ 0.11163,\ 0.78328,\ 0.39125,\ 0.24263,\\
&0.40548,\ 0.09683,\ 0.13249,\ 0.94571,\ 0.95984)
\end{aligned}$$

we define $\theta^{(1)}(t)$ as the solution of equation (B.3) corresponding to this $\theta(0)$ and to the sequence of indices given by $\bar{i}^{(1)} = (4, 4, \ldots)$.

It results that $\theta^{(1)}_{89}(t) = 0.914181$ for every $t \geq 0$ and the mean value of $\theta^{(1)}(t)$ is around 0.847197 for $t = 500$ and 0.904391 for $t = 1000$. The solution is slowly converging to $e\theta^{(1)}_{89}(0)$.

If we consider, instead, the sequence $\bar{i}^{(2)} = (4, 4, 7, 4, \ldots, 4, \ldots)$ starting always from $\theta(0)$ we obtain a new vector $\theta^{(2)}(t)$ such that $\theta_{89}^{(2)}(2) = 0.914181$, whereas $\theta_{89}^{(2)}(t) = 0.512462$, for every $t \geq 3$, and its mean value is around 0.511357 for $t = 1000$: the solution is now slowly converging to $e\theta_{89}^{(2)}(3)$.

Finally, if we consider the generic sequence

$$\bar{i}^{(3)} = (1,2,8,5,1,10,1,1,5,8,4,5,1,3,3,10,\ldots),$$

starting from the same $\theta(0)$, after 9 steps the system has almost reached a global consensus.

In fact the mean value of the solution $\theta^{(3)}(9)$ is around 0.496494 and its minimum and maximum values are 0.496466 and 0.496533 and after other 7 steps the gap between the minimum and maximum value of $\theta^{(3)}(16)$ becomes $4 \cdot 10^{-8}$, while the mean value remains the same. The generic product presents a second spectral radius which is considerably smaller than $\rho_2(F_4)$.

Consider, in fact, that the family $\widetilde{\mathscr{F}}$, obtained removing F_4 from $\mathscr{F}$, has F_3 of $\mathscr{F}$ as candidate s.m.p. with $\rho_2(F_3) = 0.517675$. This result gives us a hint to understand the rapid convergence of $\theta(t)$ for every randomly generated sequence of indices $\bar{i}$.

We plan to study in a future work the Lyapunov exponent and the joint spectral subradius of the set $\mathscr{F}$ [90, 114].

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