Assigned: Monday Apr. 12, 2010
Due: Wed Apr. 21, 2010

MATH 442: Homework 8
Spring 2010

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- At the beginning of each problem, clear state the problem (with assumptions) and what method or approach you intend to use to solve the problem.
- Clear mark your answer.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

1. Consider heat flow in a 1D rod without sources and non-constant thermal properties. Assume that the temperature is zero at $x = 0$ and $x = L$. Suppose that $c\rho_{\min} \le c(x)\rho(x) \le c\rho_{\max}$ and $K_{\min} \le K_0(x) \le K_{\max}$. Obtain an upper and (nonzero) lower bound on the slowest exponential rate of decay of the temperature.

2. Consider the boundary value problem

$$\begin{aligned} \phi'' + \lambda\phi &= 0 \\ \phi(0) - \phi'(0) &= 0 \\ \phi(1) + \phi'(1) &= 0. \end{aligned}$$

   (a) Using the Rayleigh quotient, show that $\lambda > 0$.

   (b) Show that
$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$
   Determine the eigenvalues graphically. Estimate the large eigenvalues.

3. Consider
$$\frac{d}{dx}\left((1+x)^2\frac{d\phi}{dx}\right) + \lambda(1+x)\phi = 0$$
   subject to $\phi(0) = 0$ and $\phi(1) = 0$.

   (a) Using the formula derived in class (also in the book), find the asymptotic expression for the eigenvalues when $\lambda \gg 1$.

   (b) Using the formula derived in class (also in the book), find the asymptotic expression for the eigenfunctions when $\lambda \gg 1$.

   (c) In MATLAB plot the asymptotic expression for the eigenfunctions for $n = 1$, $n = 2$, $n = 3$, and $n = 4$.

4. Consider for $\lambda \gg 1$

$$\frac{d^2\phi}{dx^2} + [\lambda\sigma(x) + q(x)]\,\phi = 0$$

subject to $\phi(0) = 0$ and $\phi(L) = 0$.

   (a) Substitute

$$\phi(x) = A(x)\,\exp\left[i\sqrt{\lambda}\int_0^x \sqrt{\sigma(s)}\,ds\right]$$

   into the ODE and determine a differential equation for $A(x)$.

   (b) Let

$$A(x) = A_0(x) + \lambda^{-1/2}A_1(x) + \cdots .$$

   Solve for $A_0(x)$ and $A_1(x)$. Verify that by only using $A_0(x)$ (i.e., neglecting the $\lambda^{-1/2}A_1(x)$ term for $\lambda \gg 1$) we get exactly the expression for $\phi(x)$ given in class. The $\lambda^{-1/2}A_1(x)$ term improves our formula.

5. The vertical displacment of a non-uniform membrane satisfies

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right),$$

where $c = c(x, y)$. Suppose that $u = 0$ on the boundary of an irregularly shaped membrane.

   (a) Show that the time variable can be separated by assuming that

$$u(x, y, t) = \phi(x, y)h(t).$$

   Show that $\phi(x, y)$ satisfies the eigenvalue problem

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0 \quad \text{with} \quad \phi = 0 \quad \text{on the boundary.}$$

   What is $\sigma(x, y)$?

   (b) Prove that eigenfunctions belonging to different eigenvalues are orthogonal.

   (c) Prove that all the eigenvalues are real.

   (d) Prove that $\lambda > 0$.

6. Using the method of eigenfunction expansions, solve the following problem:

$$\begin{aligned} \textbf{PDE:} &\quad u_t = u_{xx} + \sin(4x)e^{-2t} \\ \textbf{BCs:} &\quad u(0,t) = 1, u(\pi,t) = 0 \\ \textbf{IC:} &\quad u(x,0) = 0. \end{aligned}$$

7. Consider the following problem:

$$\begin{aligned} \textbf{PDE:} \quad & u_t = u_{xx} + (1-x)(t+1) \\ \textbf{BCs:} \quad & u(0,t) = t, \; u(1,t) = 0 \\ \textbf{IC:} \quad & u(x,0) = 0. \end{aligned}$$

   (a) Approach #1: Solve this problem without making a reduction to homogeneous boundary conditions. Call this solution $F(x,t)$.

   (b) Approach #2: Solve this problem by first making a reduction to homogeneous boundary conditions. Call this solution $G(x,t)$.

   (c) Comparison: Plot $F(x,1)$ and $G(x,1)$ on the same graph in MATLAB. Use 100 terms in each series. What do you observe?

   (d) Comparison: Let us compare these two solutions at time $t = 1$ and location $x = 0.5$. Define the following partial sums:

$$F(0.5,1) \approx F_M \equiv \cdots + \sum_{n=1}^{M} \cdots$$

$$G(0.5,1) \approx G_M \equiv \cdots + \sum_{n=1}^{M} \cdots .$$

   On log-log graph (see `loglog` command in MATLAB), plot $|F_{2M+2} - F_{2M}|$ for $M = 1$ to $M = 199$. On the same graph plot $|G_{2M+2} - G_{2M}|$ for $M = 1$ to $M = 199$. Based on these graphs, which series is converging faster? Why?