Assigned: Wednesday Mar. 3, 2010
Due: Friday Mar. 19, 2010

MATH 442: Homework 6
Spring 2010

This is due the FRIDAY after spring break!

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- At the beginning of each problem, clear state the problem (with assumptions) and what method or approach you intend to use to solve the problem.
- Clear mark your answer.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

The Wave Equation

1. Consider the boundary value problem:

$$\begin{aligned} &u_{tt} + u_{xxxx} + \delta u_t + ku = 0 \quad \text{in} \quad 0 < x < \pi \\ &u(0,t) = u(\pi,t) = 0 \\ &u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \end{aligned}$$

   where $\delta, k > 0$ are known constants, and δ is *small*. This equation, which describes the vertical motion of a beam of length π with hinged ends, is called the *beam equation*. Use separation of variables to find the general solution of this equation. (**HINT:** you do not need to check the three cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ separately. Only one case yields nonzero solution; you can simply focus on that case.)

2. Consider the following wave equation:

$$\begin{gathered} u_{tt} = u_{xx} \quad \text{for} \quad 0 < x < 1, t > 0 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = x^2(1-x), \;\; u_t(x,0) = 0. \end{gathered}$$

   (a) Find the full solution. (**HINT:** the solution should not be in terms of infinite sums.)

   (b) In MATLAB, plot the solution $u(x,t)$ at $t = 0$ and at several late r times.

   (c) In MATLAB, plot $u(0.5,t)$ versus t.

3. Show that the solution to the initially unperturbed wave equation,
$$u_{tt} = c^2 u_{xx}$$
$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$
$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = g(x),$$
is
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi,$$
where $G(x)$ is the odd extension of $g(x)$. (**HINT:** make use the separation of variables solution that we computed in class.)
What are $F(x)$ and $G(x)$?

Sturm-Liouville

4. Consider the non-Sturm-Liouville differential equation
$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\,\phi = 0.$$
Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:
$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\,\phi = 0.$$
Given $\alpha(x)$, $\beta(x)$, and $\gamma(x)$, what are $p(x)$, $\sigma(x)$, and $q(x)$.

5. Consider the eigenvalue problem
$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0 \quad \text{with} \quad \phi(1) = \phi(b) = 0.$$
   (a) Use the result from the previous problem to put this in Sturm-Liouville form.
   (b) Using the Rayleigh quotient, show that $\lambda \geq 0$.
   (c) Solve this equation subject to the boundary conditions and determine the eigenvalues and eigenfunctions. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.
   (d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.
   (e) Show that the n^{th} eigenfunction has $n-1$ zeros in $1 < x < b$.

6. Consider
$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u,$$
where c, ρ, K_0, and α are functions of x, subject to
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x).$$
Assume that the appropriate eigenfunctions are known.

(a) Using the Rayleigh quotient, show that the eigenvalues are positive if $\alpha < 0$.

(b) Solve the initial value problem.

(c) Discuss the limit as $t \to \infty$.

7. Consider the fourth-order linear differential operator:

$$\mathcal{L} = \frac{d^4}{dx^4}.$$

(a) Show that $u\mathcal{L}(v) - v\mathcal{L}(u)$ is an exact differential.

(b) Evaluate $\int_0^1 [u\mathcal{L}(v) - v\mathcal{L}(u)]\, dx$ in terms of the boundary data for any function u and v.

(c) Show that $\int_0^1 [u\mathcal{L}(v) - v\mathcal{L}(u)]\, dx = 0$ if u and v are any two functions satisfying the boundary conditions

$$\phi(0) = \phi(1) = 0$$
$$\phi'(0) = \phi''(1) = 0.$$

(d) For the eigenvalue problem (using the boundary conditions from part (c))

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0,$$

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

(e) Show that the eigenvalues in part (d) satisfy $\lambda \leq 0$. Is $\lambda = 0$ an eigenvalue?