Assigned: Monday Feb. 22, 2010
Due: Friday Feb. 26, 2008

MATH 442: Homework 5
Spring 2010

Your Mid Term is on March 1st!

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

More Fourier Series

1. (a) Find the Fourier series for

$$f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ 1 - x^2 & \text{if } 0 < x \le 1 \end{cases}$$

   defined on the interval $-1 \le x \le 1$.

   (b) In MATLAB, plot the first 20 terms and the first 200 terms of the sine series in the interval $-3 \le x \le 3$.

   (c) To what value does the series converge at $x = 0$?

2. The integration-by-parts formula

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx$$

   is valid for functions $u(x)$ and $v(x)$ are continous and have a continuous first derivative.

   Assume that $u(x)$, $v(x)$, $u'(x)$, and $v'(x)$ are all continuous for $a \le x < c$ and $c < x \le b$, but that all may have a jump discontinuity at $x = c$.

   (a) Derive an expression for $\int_a^b u(x)v'(x)\,dx$ in terms of $\int_a^b u'(x)v(x)\,dx$.

   (b) Show that the expression in (a) reduces to the standard integration-by-parts formula if $u(x)$ and $v(x)$ are continuous across $x = c$ (even if $u'(x)$ and $v'(x)$ are discontinuous across $x = c$).

3. The Fourier series of the function $f(x) = \cos(ax)$ on the interval $[-\pi, \pi]$, when a is not an integer, is given by

$$\cos(ax) = \frac{2a\sin(a\pi)}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2 - a^2}\cos(nx)\right] \quad \text{for} \quad -\pi \le x \le \pi.$$

(a) Differentiate both sides of this equation with respect to x, differentiating the series term by term, to find the Fourier series for $\sin(ax)$:

$$\sin(ax) = -\frac{2\sin(a\pi)}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2-a^2}\sin(nx)\right] \quad \text{for} \quad -\pi < x < \pi.$$

(b) Explain why this method for computing the Fourier series is valid.

(c) If you know the Fourier series for $\sin(ax)$ given in (a), why can you not differentiate it term by term with respect to x to derive the Fourier series for $\cos(ax)$.

(d) Now consider the Fourier series of $\sin(ax)$ given in (a) as known. Explain why it can be integrated term by term to get the Fourier expansion of $\cos(ax)$.

(e) Carry out this term by term integration from 0 to x, and use it to show that

$$A_0 = \frac{\sin(a\pi)}{a\pi} = 1 + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-a^2}.$$