Assigned: Monday Feb. 15, 2010
Due: Wednesday Feb. 22, 2010

MATH 442: Homework 4
Spring 2010

Your Mid Term is on March 1st!

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- At the beginning of each problem, clear state the problem (with assumptions) and what method or approach you intend to use to solve the problem.
- Clear mark your answer.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

Properties of Laplace's Equation

1. Using the maximum principle for Laplace's equation, prove that the solution of Poisson's equation, $\nabla^2 u = g(\vec{x})$, subject to $u = f(\vec{x})$ on the boundary, is unique.

2. Show that the "backward" heat equation
$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial^2 u}{\partial x^2}$$
subject to $u(0,t) = u(L,t) = 0$ and $u(x,0) = f(x)$, is *not* well-posed. (**HINT:** Solve the problem with separation of variables; actually the solution is the same as heat equation, but with κ replaced by $-\kappa$. Then show that if the initial conditions are changed by an arbitrarily small amount, for example,
$$f(x) \Longrightarrow f(x) + \frac{1}{m}\sin\left(\frac{m\pi x}{L}\right)$$
where m is an arbitrarily large integer, then the solution $u(x,t)$ changes by a large amount.)

Separation of Variables:

3. Consider the following homogeneous PDE and BCs:
$$\begin{gathered} u_t = u_{xx} \quad \text{in } 0 < x < 1 \\ u(x,0) = f(x) \\ u(0,t) = 0, \;\; u(1,t) + u_x(1,t) = 0 \end{gathered}$$

(a) Make the substitution $u(x,t) = \phi(x)\,G(t)$, separate variables, and find the equations for $\phi(x)$ and $G(t)$. Be sure to include the boundary conditions appropriately.

(b) Show that only $\lambda > 0$ produces non-trivial solutions. If $\lambda > 0$, find the equation satisfied by the eigenvalues. Unlike the previous examples that we have seen, you will *not* be able to solve for the eigenvalues explicitly.

(c) Write the equation that the eigenvalues satisfy as

$$F(\lambda) = 0.$$

Find approximate values for the first four eigenvalues by using the MATLAB function `fzero` to compute the roots of $F(\lambda)$. You will need to provide `fzero` with initial guesses; create a MATLAB plot of $F(\lambda)$ to get an idea where the roots are (Make sure you turn this plot in with the rest of the assignment).

Fourier Series

4. (a) Find the Fourier *sine* series for $f(x) = 1 - x^2$ defined on the interval $0 \le x \le 1$.

   (b) In MATLAB, plot the first 20 terms and the first 200 terms of the sine series in the interval $-3 \le x \le 3$.

   (c) To what value does the series converge at $x = 0$?

5. (a) Find the Fourier *cosine* series for $f(x) = 1-x^2$ defined on the interval $0 \le x \le 1$.

   (b) In MATLAB, plot the first 20 terms and the first 200 terms of the cosine series in the interval $-3 \le x \le 3$.

   (c) To what value does the series converge at $x = 0$?