Assigned: Monday Jan 18, 2010
Due: Friday Jan 29, 2010

MTH 442: Homework 2
Spring 2010

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

1. Consider the polar coordinates

$$x = r\cos(\theta)$$
$$y = r\sin(\theta).$$

   (a) Since $r^2 = x^2 + y^2$, show that $\frac{\partial r}{\partial x} = \cos(\theta)$, $\frac{\partial r}{\partial y} = \sin(\theta)$, $\frac{\partial \theta}{\partial y} = \frac{\cos(\theta)}{r}$, and $\frac{\partial \theta}{\partial x} = -\frac{\sin(\theta)}{r}$.

   (b) Show that $\hat{\mathbf{r}} = \cos(\theta)\,\hat{\mathbf{i}} + \sin(\theta)\,\hat{\mathbf{j}}$ and $\hat{\boldsymbol{\theta}} = -\sin(\theta)\,\hat{\mathbf{i}} + \cos(\theta)\,\hat{\mathbf{j}}$.

   (c) Using the chain rule, show that $\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta}$ and hence $\nabla u = \hat{\mathbf{r}}\,\frac{\partial u}{\partial r} + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial u}{\partial \theta}$.

   (d) If $\vec{A} = A_1\hat{\mathbf{r}} + A_2\hat{\boldsymbol{\theta}}$, show that $\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_1) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_2)$, since $\partial\hat{\mathbf{r}}/\partial\theta = \hat{\boldsymbol{\theta}}$ and

   $\partial\hat{\boldsymbol{\theta}}/\partial\theta = -\hat{\mathbf{r}}$ follows from part (b).

   (e) Show that $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

2. **Heat Equation with Circular Symmetry.** Assume that the temperature is circularly symmetric: $u = u(r,t)$, where $r^2 = x^2 + y^2$. Consider any circular annulus $a \le r \le b$.

   (a) Show that the total heat energy is $2\pi\int_a^b c\rho u r\,dr$.

   (b) Show that the flow of heat energy per unit time out of the annulus at $r = b$ is

$$\phi = -2\pi b K_0 \frac{\partial u}{\partial r}\bigg|_{r=b}.$$

   A similar results holds at $r = a$.

   (c) Assuming the thermal properties are spatially homogeneous, use parts (a) and (b) to derive the circularly symmetric heat equation without sources:

$$\frac{\partial u}{\partial t} = \frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

   (d) Find the equilibrium temperature distribution inside the circular annulus $a \le r \le b$ if the outer radius is insulated and the inner radius is at temperature T.

3. If Laplace's equation is satisfied in 3D, show that

$$\oiint \nabla u \cdot \hat{\mathbf{n}}\, dS = 0\,.$$

for any closed surface. (**HINT:** use the divergence theorem.) Give a physical interpretation of this result in the context of heat flow.

4. If $\mathcal{L}$ is a linear operator, show that $\mathcal{L}\left(\sum_{n=1}^{M} c_n u_n\right) = \sum_{n=1}^{M} c_n \mathcal{L}(u_n)$. Use this result to show that the principle of superposition may be extended to any finite number of homogenous solutions.

5. Evaluate (be careful if $n = m$)

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)\, dx$$

for $n > 0$ and $m > 0$. Use the trigonometric identity

$$2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b).$$

6. **Separation of Variables.** By using $u(x,t) = X(x)T(t)$ or $u(x,y,t) = X(x)Y(y)T(t)$, separate the following PDEs into two or three ODEs for X and T or X, Y, and T. The parameters c and k are constants. You do not need to solve the equations. **NOTE:** one of the equations cannot be separated. Inidicate this when you discover that equation.

   (a) $u_{tt} = (xu_x)_x$
   (b) $u_{tt} = c^2 u_{xx}$
   (c) $u_t = \kappa\left(u_{xx} + u_{yy}\right)$
   (d) $u_t = \kappa\left(yu_x + u_y\right)$
   (e) $u_t + cu_x = \kappa u_{xx}$
   (f) $u_t = \kappa\left(yu_x + xu_y\right)$