Assigned: Monday Jan 11, 2010
Due: Monday Jan 18, 2010

MTH 442: Homework 1
Spring 2010

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

1. Solve each of the following ODEs for $y(x)$ with initial conditions $y(0) = 0$ and $y'(0) = 1$:
   (a) $y'' + \frac{\pi}{2}y = 0$
   (b) $y'' + \pi y' + \frac{\pi}{2}y = 0$
   (c) $y'' - \pi y = x^2$.

2. Find all of the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$.

3. A PDE can be written in differential operator notation $\mathcal{L}(u) = f$, where $\mathcal{L}$ is the differential operator, u is the unknown function, and f is the right-hand side function. For each of the following PDEs, determine the linear operator and the right-hand side function, the order of the PDE, whether it is linear or nonlinear, and whether it is homogeneous or nonhomogeneous:
   (a) $u_{xxx} + u_{yyy} - xy = 2$
   (b) $u_t - au_{xx} = 0$
   (c) $u_t - uu_x = \epsilon u_{xx}$

4. The following convection-diffusion-decay equation appears in many physical applications:
$$u_t = Du_{xx} - cu_x - \lambda u\,.$$
   Show that this equation can be transformed into a diffusion equation for $w(x, t)$ by applying the transformation
$$u(x, t) = w(x, t)e^{\alpha x - \beta t}\,.$$
   HINT: You will only obtain a diffusion equation for $w(x, t)$ with an appropiate choice for the constants α and β in terms of the constants D, c, and λ. Determine the choice for α and β that produces a diffusion equation for $w(x, t)$.

5. Consider the heat equation:

$$u_t = \big(K_0(x)\, u_x\big)_x$$

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 1\,, \end{aligned}$$

where $K_0(x) = (x+1)^2$.

   (a) Determine the steady-state solution.

   (b) Plot the steady-state solution in MATLAB. Always clearly label all plots.

6. Consider the heat equation:

$$u_t = \big(K_0(x)\, u_x\big)_x + Q(t)$$

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 1\,, \end{aligned}$$

where $K_0(x) = (x+1)^2$.

   (a) Under what condition on the heat source $Q(t)$ does a steady-state solution exist for this problem? Clearly explain your answer and give a physical interpretation of this result.

   (b) Under this condition, determine the steady-state solution.

7. Consider the function:

$$u(x,t) = \left(\frac{40}{\pi}\right) \sin(\pi x)\, e^{-\pi t}\,.$$

Plot this function in MATLAB over the domain $(x,t) \in [0,1] \times [0,1]$ using the `mesh` command. Always clearly label all plots. (**HINT:** see page 4 of the "Introduction to Plotting with MATLAB" guide.)